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ELEMENTS
OF
TRIGONOMETRY

WITH TABLES

BY

W. C. BRENKE

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF NEBRASKA

NEW YORK
THE CENTURY CO.
1917

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PREFACE

This book contains, in revised form and with some additions, the material of the chapters on trigonometry found in the author's text-book on "Advanced Algebra and Trigonometry." The theory of the subject is here presented in a manner as direct and elementary as possible, and much attention is given to the applications, which are brought out in numerous examples and exercises. These cover a sufficient range to give the student in technical courses a working knowledge of trigonometry as a tool.

Great care has been taken to make the illustrations instructive, and a large amount of graphic work is called for in the exercises throughout the text.

The presentation of spherical trigonometry, with applications to the terrestrial globe and to practical astronomy, has been reduced to the space of 17 pages, which will be found to contain a fair introduction to the subject. If this is to be included with plane trigonometry in a three-hour course, the work in identities and in the numerical solution of triangles should be reduced to a minimum.

Four-place tables are appended to the text, in the belief that they are sufficient for a large range of practical applications, and, because of their brevity, are less confusing to beginners and enable them to get results in less time than would be the case with more extensive tables. The explanation of the table of logarithms is made on the assumption that the student has no knowledge of the subject of logarithms.

Answers are given to nearly all of the odd-numbered exercises. This gives the instructor the liberty of assigning exercises with or without given answers, as may seem best.

For the convenience of the student a protractor is provided and will be found inside the back cover.

For a five-hour course in Algebra and Trigonometry combined, this text may be had bound in one volume with the author's text-book on Advanced Algebra.

W. C. Brenke.

Lincoln, Nebraska,
March, 1917.

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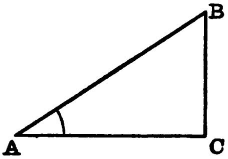
ELEMENTS OF TRIGONOMETRY

CHAPTER I

THE TRIGONOMETRIC FUNCTIONS

1. The Sine of an Angle. — Let us consider a given acute angle A, and suppose that we have constructed a right triangle ABC, containing angle A. Suppose the lengths of AB and CB to be measured, and the value of their ratio, $CB \div AB$, to be calculated. This ratio is called *the sine of angle A*, and is defined as follows:

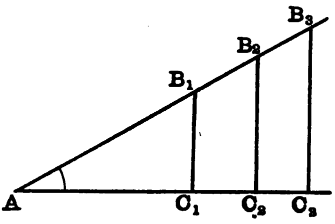


Given any acute angle A, form a right triangle with $\angle A$ as one of its angles. Then

$$\textit{sine of } \angle A = \frac{\textit{opposite side}}{\textit{hypotenuse}}.$$

The abbreviation for *sine of* $\angle A$ is $\sin A$.

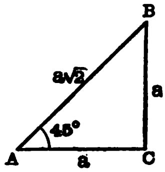
This quantity, that is $\sin A$, has the same value for all right triangles from which it may be obtained. For, all right triangles containing a given angle A are mutually equi-angular, and hence similar. Therefore the ratios of corresponding sides are equal. The figure shows several such triangles placed with their angles A superposed. Then the ratios



$$\frac{\textit{side opposite } \angle A}{\textit{hypotenuse}} = \frac{C_1B_1}{AB_1} = \frac{C_2B_2}{AB_2} = \frac{C_3B_3}{AB_3}$$

are all equal, and, by definition, determine the value of $\sin A$.

Therefore the sine of an angle is a number whose value is fixed by the size of the angle, independently of the size of the triangle from which the ratio is formed. The size of an angle alone is sufficient to determine the value of its sine, and so we say:



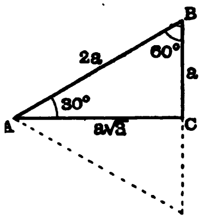
The sine of an angle is a function of the angle.

To illustrate, suppose $\angle A = 45°$. Construct any *isosceles* right triangle and let a be the length of each of the equal sides. Then the length of the hypotenuse is $a\sqrt{2}$. Each acute angle is 45°. Then

$$\sin 45° = \frac{\textit{opp. side}}{\textit{hypotenuse}} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = 0.707+.$$

The actual length, a, of the side of the triangle cancels out in the ratio. We might just as well have taken the three sides of the triangle to be 1, 1, $\sqrt{2}$, respectively.

Again, suppose $\angle A = 30°$ or $60°$. We can form a right triangle with acute angles 30° and 60° by bisecting an equilateral triangle, as shown in the figure. If each side of the original triangle is $2a$, then, after bisection, the side opposite 30° is a, and that opposite 60° is $a\sqrt{3}$. Then by definition,



$$\sin 30° = \frac{\textit{opp. side}}{\textit{hypotenuse}} = \frac{a}{2a} = \frac{1}{2} = 0.5.$$

$$\sin 60° = \frac{\textit{opp. side}}{\textit{hypotenuse}} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2} = 0.866+.$$

Here again, a cancels out in the ratios, and we might just as well have marked the three sides of the right triangle 1, $\sqrt{3}$, 2, respectively.

2. The Six Trigonometric Functions. — The ratio of the side opposite the angle to the hypotenuse is merely one of **six possible ratios** which may be formed from the three sides of any right triangle. Hence associated with every acute angle there are six ratios, or six abstract numbers, whose values depend merely on the magnitude of the angle. They are called the **six trigonometric ratios,** or **trigonometric functions of the angle,** and are named as follows:

$$\text{sine of } \angle A = \sin A = \frac{\text{opposite side}}{\text{hypotenuse}}.$$

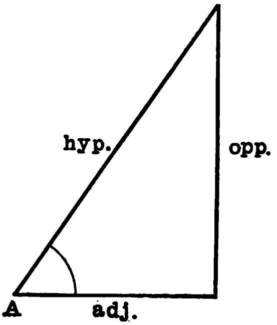
$$\text{cosine of } \angle A = \cos A = \frac{\text{adjacent side}}{\text{hypotenuse}}.$$

$$\text{tangent of } \angle A = \tan A = \frac{\text{opposite side}}{\text{adjacent side}}.$$

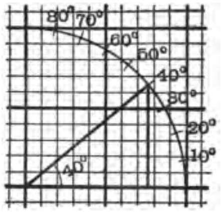
$$\text{cosecant of } \angle A = \csc A = \frac{\text{hypotenuse}}{\text{opposite side}}.$$

$$\text{secant of } \angle A = \sec A = \frac{\text{hypotenuse}}{\text{adjacent side}}.$$

$$\text{cotangent of } \angle A = \cot A = \frac{\text{adjacent side}}{\text{opposite side}}.$$



Exercise. With the aid of a protractor (see inside of back cover) construct triangles containing angles 10°, 20°, and so on at intervals of 10° up to 80°. By measuring the sides, and dividing, calculate the six functions of each of these angles to two decimal places and tabulate results. Construct all the triangles on a single diagram, such as is shown in the figure. Check results by comparison with the table which follows.

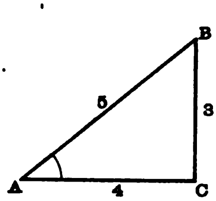


3. Table of the Trigonometric Functions. —

Angle.	Sin.	Cos.	Tan.	Cot.	Sec.	Csc.
0°						
5	0.087	0.996	0.087	11.430	1.004	11.474
10	0.174	0.985	0.176	5.671	1.015	5.759
15	0.259	0.966	0.268	3.732	1.035	3.864
20	0.342	0.940	0.364	2.748	1.064	2.924
25	0.423	0.907	0.466	2.144	1.103	2.366
30	0.500	0.866	0.577	1.732	1.155	2.000
35	0.574	0.819	0.700	1.428	1.221	1.743
40	0.643	0.766	0.839	1.192	1.305	1.556
45	0.707	0.707	1.000	1.000	1.414	1.414
50	0.766	0.643	1.192	0.839	1.556	1.305
55	0.819	0.574	1.428	0.700	1.743	1.221
60	0.866	0.500	1.732	0.577	2.000	1.155
65	0.906	0.423	2.145	0.466	2.366	1.103
70	0.940	0.342	2.748	0.364	2.924	1.064
75	0.966	0.259	3.732	0.268	3.864	1.035
80	0.985	0.174	5.671	0.176	5.759	1.015
85	0.996	0.087	11.430	0.087	11.474	1.004
90						

4. Given one function, to determine the other functions. — When a function of an acute angle is given, the angle may be constructed by writing the given function as a fraction, and constructing a right triangle, two of whose sides are the numerator and denominator of this fraction respectively, or equal multiples of these quantities. Also, since the third side of the triangle can be calculated from the other two, all the other functions of the angle may be found when one function is given.

Example 1. $\tan A = \frac{3}{4}\ \left(= \frac{\text{opp. side}}{\text{adj. side}}\right).$



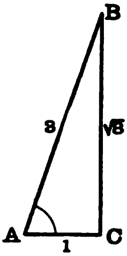
Lay off $AC = 4$ and $CB = 3$, $\perp AC$.

Then $AB = \sqrt{4^2 + 3^2} = 5.$

Hence $\sin A = \frac{3}{5};\ \cos A = \frac{4}{5};$

$\csc A = \frac{5}{3};\ \sec A = \frac{5}{4};\ \cot A = \frac{4}{3}.$

Scaling off the angle with a protractor, we have $A = 37°$. By taking from the table the angle whose tangent is .75 we have $A = 37°$ as before.



Example 2. $\sec A = 3\ \left(= \frac{3}{1} = \frac{\text{hyp.}}{\text{adj. side}}\right).$

Lay off $AC = 1$. With A as center and radius $= 3$, strike an arc to cut the $\perp$ drawn to AC at C. This determines the point B.

The solution may now be completed as in example 1

Another method of constructing the triangle in this example is to calculate CB first, and then to proceed as in example 1.

5. Exercises. Determine the angle (approximately) and the remaining functions, when

1. $\sin A = \frac{12}{13}.$

2. $\sin A = \frac{2}{5}.$

3. $\sin A = 0.4.$

4. $\cos A = \frac{2}{5}.$

5. $\cos A = \frac{1}{3}.$

6. $\tan A = \frac{5}{3}.$

7. $\tan A = 3.$

8. $\tan A = \sqrt{3}.$

9. $\cot A = 1.$

10. $\cot A = 2.5.$

11. $\sec A = 2.$

12. $\csc A = \frac{3}{2}.$

13. $\cos A = 0.3.$

14. $\csc A = 2.5.$

15. $\tan A = 10.$

16. Show that the equation $\sin A = 2$ is impossible.

17. Show that the equation $\cos A = 1.1$ is impossible.

18. Show that the equation sec $A = \frac{1}{2}$ is impossible.

19. Show that the equation csc $A = .9$ is impossible.

When A is an acute angle show that,

20. sin A lies between 0 and 1.

21. cos A lies between 0 and 1.

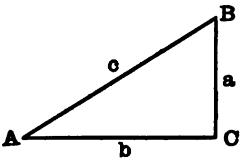
22. sec A and csc A are always greater than 1.

23. tan A and cot A may have any value from 0 to ∞.

6. Functions of Complementary Angles. — Since the sum of the two acute angles of a right triangle is 90°, they are complementary.

By definition, we have

$$\sin B = \frac{\text{opp. side}}{\text{hyp.}} = \frac{b}{c} = \cos A.$$



By considering the other functions and tabulating results we have:

$\sin B = \cos A;$ $\quad \tan B = \cot A;$ $\quad \csc B = \sec A;$
$\cos B = \sin A;$ $\quad \cot B = \tan A;$ $\quad \sec B = \csc A.$

Complementary Functions, or Co-functions. — The co-sine is called the complementary function to the sine and conversely. Similarly tangent and co-tangent are mutually complementary, and secant and co-secant.

The preceding equations are now all contained in the following

Rule: *Any function of an acute angle is equal to the co-function of the complementary angle.*

Exercise. Verify this rule when $A = 30°, 45°$, and $60°$. See also the table on p. 3.

7. Application of the Trigonometric Functions to the Solution of Right Triangles. — When two parts of a right triangle are known, exclusive of the right angle, the triangle may be constructed and the remaining parts determined graphically. By the aid of tables of the trigonometric functions, the unknown parts may also be calculated.

Rule: *When two parts of a right triangle are given (the rt. ∠ excepted) and a third part is required, write down that equation of* (**2**) *which involves the two given parts and the required part. Substitute in it the values of the given parts, and solve for the required part.*

An exceptional case arises when two sides are given and the third side is required. In this case we may use the formula $a^2 + b^2 = c^2$. It will usually be better, however, unless the given sides are represented by simple numbers, to solve for one of the angles first, and then to obtain the third side from this angle and one of the given sides.

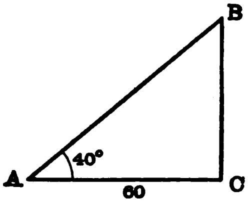
Example. In △ ABC, given $A = 40°$, $C = 90°$, and $b = 60$. Find the other parts of the triangle.

To get B, we have $B = 90° - A = 50°$.

To get a, take $\frac{a}{b} = \tan A$ or $a = b \tan A$.

Finally, c is determined from

$$\frac{b}{c} = \cos A \quad \text{or} \quad c = \frac{b}{\cos A} = b \sec A.$$



From the table of (3), $\tan 40° = 0.839$
and $\sec 40° = 1.305$.
Hence $a = 60 \times 0.839 = 50.340$
and $c = 60 \times 1.305 = 78.300$.

As a check, we should have $\frac{a}{c} = \cos B$ or $\frac{50.340}{78.300} = 0.643$.

8. Exercises.

Determine the unknown parts of right triangle ABC, C being 90°, from the parts given below. Check results by graphic solution and by a check formula containing the unknown parts. Use the table of (3).

1. $A = 35°$, $a = 100$.
2. $A = 65°$, $b = 150$.
3. $A = 48°$, $c = 75$.
4. $B = 33°$, $c = 50$.
5. $B = 58°$, $b = 750$.
6. $B = 15°$, $a = 0.15$.
7. $A = 50°$, $c = 0.045$.
8. $B = 80°$, $c = 1.25$.
9. $B = 52°$, $a = 16\frac{2}{3}$.
10. $A = 25°$, $b = \frac{1}{25}$.

11. Find the length of chord subtended by a central angle of 110° in a circle of radius 50 ft. (First find the half-chord.)

12. Find the central angle subtended by a chord of 90 ft. in a circle of radius 200 ft.

13. Find the radius of the circle in which a chord of 120 ft. subtends an angle of 70°.

14. Find the length of side of a regular decagon inscribed in a circle of radius 300 ft.

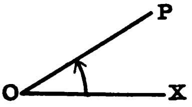
15. Find the length of side of a regular pentagon circumscribed about a circle of radius 200 ft.

16. From a point in the same horizontal plane as the foot of a flag-pole, and 200 ft. from it, the angle of elevation of the top is 20°. How high is the pole?

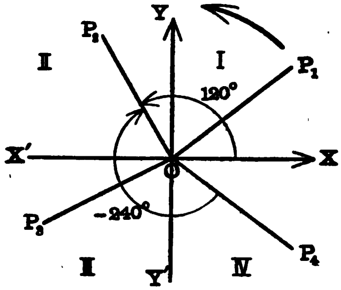
17. A vertical pole 35 ft. high casts a shadow 50 ft. long on level ground Find the altitude of the sun.

9. Angles of any Magnitude, Positive or Negative. — Consider ∠ XOP (figure) as generated by a moving line which rotates about O from the position OX to the position OP.

Divide the plane into four quadrants (I, II, III, and IV in the figure below) by means of two rectangular axes XX' and YY'.



Quadrant I is that covered by a line rotating from OX to OY in the direction of the curved arrow, the angle turned through being 90°. Let a moving line start from the position OX and rotate into the positions OP_1, OP_2, OP_3, and OP_4 successively, thus generating the angles XOP_1, XOP_2, XOP_3, and XOP_4 respectively.



OX is called the **initial line,** and OP_1 the **terminal line** of the angle XOP_1, and similarly for any other angle.

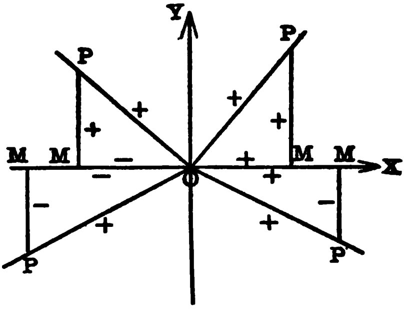
An angle is **positive** when the generating line rotates **counterclockwise** (in the direction of the curved arrow in the figure), **negative** when the generating line moves **clockwise.**

The **quadrant of an angle** is that quadrant in which its terminal line lies. The angle is said to lie in this quadrant.

The initial line OX, and any terminal line, as OP_2, may always be considered to form two angles numerically less than 360°, as $+120°$ and $-240°$ in the figure.

When the moving line rotates from OX through more than one complete revolution, an angle greater than 360° is generated. Thus a rotation in the positive direction (positive rotation) through $1\frac{1}{3}$ revolutions generates an angle of 480°, lying in the second quadrant; a negative rotation through $2\frac{1}{6}$ revolutions generates an angle of $-780°$, lying in the fourth quadrant.

10. The Trigonometric Functions of any Angle. — Let XOP be any angle, and P any point in its terminal line. (The four possible cases are here shown in the figure, according to the quadrant of the angle.) With respect to any point P, three lines are to be noted, namely:



(1) The perpendicular MP, from the initial line to the point;
(2) The segment OM measured along the initial line;
(3) The distance OP, from the origin to the point P.

These lines are named as follows:

MP is called the *ordinate* of the point P;
OM is called the *abscissa* of the point P;
OP is called the *distance* of the point P.

Moreover, signs are attached to the *ordinate* and *abscissa*, to indicate the quadrant in which P lies. Thus:

Ordinate MP is *plus* when measured *upward, minus* when measured *downward;*

Abscissa OM is *plus* when measured *to the right, minus* when measured *to the left;*

Distance OP is always taken *plus.*

We now define the functions of angle XOP, in whatever quadrant it may be, as follows:

$$\sin XOP = \frac{\text{ordinate (of } P)}{\text{distance (of } P)}; \qquad \csc XOP = \frac{\text{distance}}{\text{ordinate}};$$

$$\cos XOP = \frac{\text{abscissa}}{\text{distance}}; \qquad \sec XOP = \frac{\text{distance}}{\text{abscissa}};$$

$$\tan XOP = \frac{\text{ordinate}}{\text{abscissa}}; \qquad \cot XOP = \frac{\text{abscissa}}{\text{ordinate}}.$$

For an acute angle these definitions agree with those of (**2**), the only difference being that opposite side is now called ordinate, adjacent side is called abscissa, and hypotenuse is called distance.

According to the above definitions we have the following

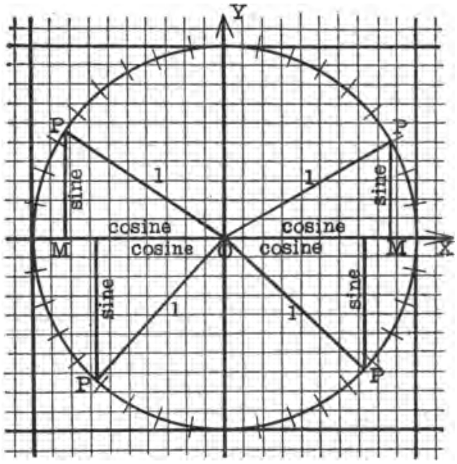
TABLE OF SIGNS OF THE TRIGONOMETRIC FUNCTIONS

Quadr.	sin.	cos.	tan.	cot.	sec.	csc.
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−

Let the student verify carefully the signs in this table. He should be prepared to state instantly the sign of any function in any quadrant.

Observe that in the first quadrant all the functions are positive; in the other quadrants a function and its reciprocal are positive, the remaining four negative.

11. Approximate Values of the Functions of any Angle. — If in the last figure the distances OP had been taken all of the same length, all the points P would lie on the circumference of a circle with center at O.



Let us draw a circle with O as center and unit radius (figure; 1 = 10 small divisions). Then for any angle XOP we have

$$\sin XOP = \frac{MP}{1} = MP,$$

$$\cos XOP = \frac{OM}{1} = OM.$$

Thus the figure shows sin 30° = .5 and cos 30° = .86; sin 147° = .56 and cos 147° = −.83; sin 228° = −.73 and cos 228° = −.67; sin 317° = −.69 and cos 317° = .72.

By noting the values of MP at regular intervals as P moves around the circumference, a complete table of values may be constructed. See Exercise 4 of (**13**).

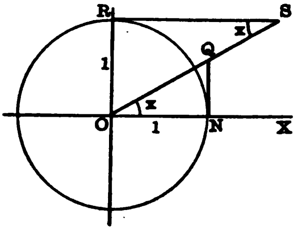
Hence approximate values of the sines and cosines of all angles may be read off directly from the figure. The other functions may be obtained by division, since $\tan XOP = \frac{MP}{OM}$, etc. They may also be constructed graphically by a method explained in the next article.

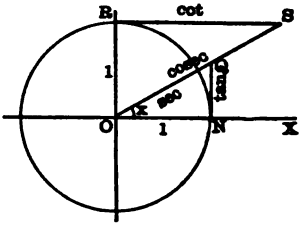
The lines MP and OM, whose lengths, measured with OP as unit of length, represent the sine and cosine of $\angle XOP$, are commonly referred to as the **line values** of these functions.

The term "*sine*," from the Latin "*sinus*" (chord), arises from the fact that if MP were doubled it would form a chord of the circle

12. Line Values of the other Functions. — We shall consider the case of acute angles only; the other cases will be of no particular interest to us.

Let x then be an acute angle, with initial line OX and terminal line OQ, NQ being drawn tangent to the circle of radius 1 and center at the vertex of angle x.





In $\triangle ONQ$:

$$\tan x = \frac{\text{opp.}}{\text{adj.}} = \frac{NQ}{1} = NQ; \qquad \sec x = \frac{\text{hyp.}}{\text{adj.}} = \frac{OQ}{1} = OQ.$$

Hence $\tan x$ is measured by a segment of a line tangent to the circle, and $\sec x$ is measured by a segment of a secant line. This explains the origin of the names of these functions.

Also, in $\triangle ORS$, where RS is a tangent line and OS is OQ produced, we have

$$\cot x = \frac{\text{adj.}}{\text{opp.}} = \frac{RS}{1} = RS; \qquad \csc x = \frac{\text{hyp.}}{\text{opp.}} = \frac{OS}{1} = OS.$$

These results are indicated in the second figure.

13. Variation of the Trigonometric Functions. — In the figure of (**11**) suppose the point P to describe the circumference of the circle in such a way that the angle XOP shall vary continuously from 0° to 360°. Let us trace the changes in the value of $\sin XOP = MP$. In the first quadrant MP, and hence $\sin XOP$, varies from 0 to $+1$, in the second from $+1$ to 0, in the third from 0 to -1, and in the fourth from -1 to 0.

Similarly $\cos XOP$ varies in the four quadrants successively from $+1$ to 0, 0 to -1, -1 to 0, and 0 to $+1$.

Consider next $\tan XOP = \frac{MP}{OM}$. When $XOP = 0°$, $MP = 0$, and $OM = 1$; hence $\tan 0° = 0$.

Now as XOP increases from 0° toward 90°, MP steadily increases toward 1, while OM steadily diminishes toward 0. Hence $\tan XOP$ increases from 0 without limit, so that we write $\tan 90° = \infty$, and say that the tangent varies from 0 to ∞ as XOP varies from 0° to 90°.

Just before $\angle XOP$ reaches 90°, $\tan XOP$ is a very large positive number; just after $\angle XOP$ has passed 90°, $\tan XOP$ is a very large *negative* number. This abrupt change in the value of the tangent when the angle passes through 90° is expressed by saying that $\tan x$ has a **discontinuity** at $x = 90°$. There is another discontinuity at 270°.

Since the three remaining functions are reciprocals of the three already considered, their variations are easily traced. Thus, $\csc XOP = \frac{1}{\sin XOP}$. Hence $\csc XOP$ varies from ∞ to 1 in the first quadrant, and from 1 to ∞ in the second. Now as XOP passes through 180°, $\csc XOP$ changes suddenly from a large positive value when the angle is a little less than 180° to a large negative value when the angle is a little more than 180°.

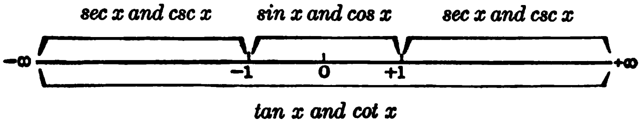
Therefore the cosecant has a **discontinuity** at 180°; $\csc 180°$ may be either $+\infty$ or $-\infty$, according to the side from which XOP approaches 180°.

In the third quadrant $\csc XOP$ is negative and varies from $-\infty$ to -1; in the fourth quadrant from -1 to $-\infty$. There is another discontinuity at 360° or 0°.

The variations of the six functions are shown in the following table.

Quadr.	sin.	csc.	cos.	sec.	tan.	cot.
I	0 to $+1$	$+\infty$ to $+1$	$+1$ to 0	$+1$ to $+\infty$	0 to $+\infty$	$+\infty$ to 0
II	$+1$ to 0	$+1$ to $+\infty$	0 to -1	$-\infty$ to -1	$-\infty$ to 0	0 to $-\infty$
III	0 to -1	$-\infty$ to -1	-1 to 0	-1 to $-\infty$	0 to $+\infty$	∞ to 0
IV	-1 to 0	-1 to $-\infty$	0 to $+1$	$+\infty$ to 1	$-\infty$ to 0	0 to $-\infty$

The range of values covered by each of the six functions is indicated in the diagram below.



Exercises.

1. Trace carefully the variations given in the above table.

2. Show that the following functions have discontinuities at the values stated: the cotangent, at 0° and 180°; the secant, at 90° and 270°.

3. Discuss the "equations," $\tan 90° = +\infty$; $\tan 90° = -\infty$. Same for $\csc 0° = +\infty$; $\csc 0° = -\infty$.

4. Draw a figure as in (**11**). Divide the circumference into 36 equal parts, and obtain from the diagram a two-place table of the six functions for every tenth degree from 0° to 360°.

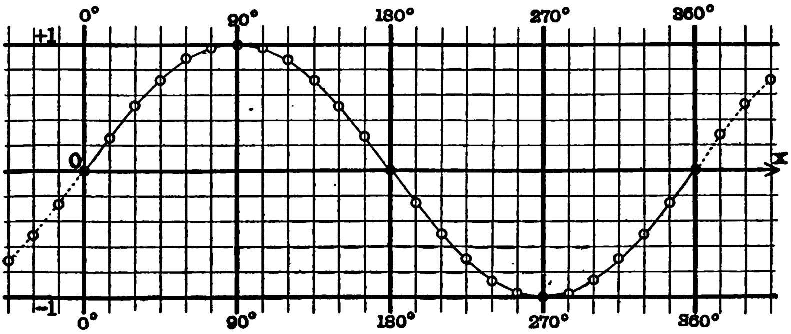
14. Graphic Representation of the Trigonometric Functions. — It is instructive to represent on a diagram such a table as that called for in the last exercise of (**13**). Let us note the values of $\sin x$ to two decimals, for values of x at intervals of 15° from $x = 0°$ to 360°.

x	$\sin x$	x	$\sin x$	x	$\sin x$
0°	0.00	135°	0.71	270°	−1.00
15°	0.26	150°	0.50	285°	−0.97
30°	0.50	165°	0.26	300°	−0.87
45°	0.71	180°	0.00	315°	−0.71
60°	0.87	195°	−0.26	330°	−0.50
75°	0.97	210°	−0.50	345°	−0.26
90°	1.00	225°	−0.71	360°	0.00
105°	0.97	240°	−0.87		
120°	0.87	255°	−0.97		

We now mark off an *angle scale*, taking any convenient length to represent a unit of angle. In the figure below the scale is such that the side of one square represents 15°.

Above (or below) each value of x we place a dot whose distance from the angle scale shall show the value of $\sin x$. Any convenient vertical distance may be chosen to represent sin 90° or 1. In the figure it is taken as the length of 5 squares.

When $\sin x$ is positive the dot is placed above the angle scale, and below when $\sin x$ is negative.



Joining the dots by a smooth curve gives us a graphic representation of the variations in the sine of an angle when the angle varies from 0° to 360°. The approximate value of the sine of any angle could be read off at once from the diagram.

The curve has a simple wave form. It is called *the graph of sin x*. By taking values of x from 360° to 720° another wave would appear, and so on indefinitely. By taking negative values of x, to the left from 0°, these waves could be continued to the left.

The graphs of the other functions can be constructed in the same manner. They are shown on the following page.

Exercise. From your table of Exercise 4, p. 12, construct the graphs of the six functions. (An interesting chart can be made by doing this work on a large sheet of paper, and using colored inks or pencils to distinguish the various curves.)

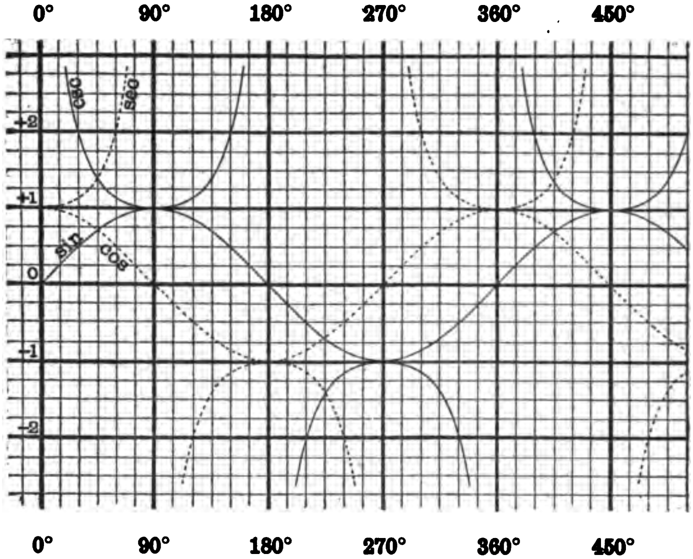
Periodicity of the Trigonometric Functions. — Since the position of the terminal line of an angle x is unchanged when the angle is increased or diminished by integral multiples of 360°, any function of x equals the same function of $x \pm n \cdot 360°$, n being an integer. That is,

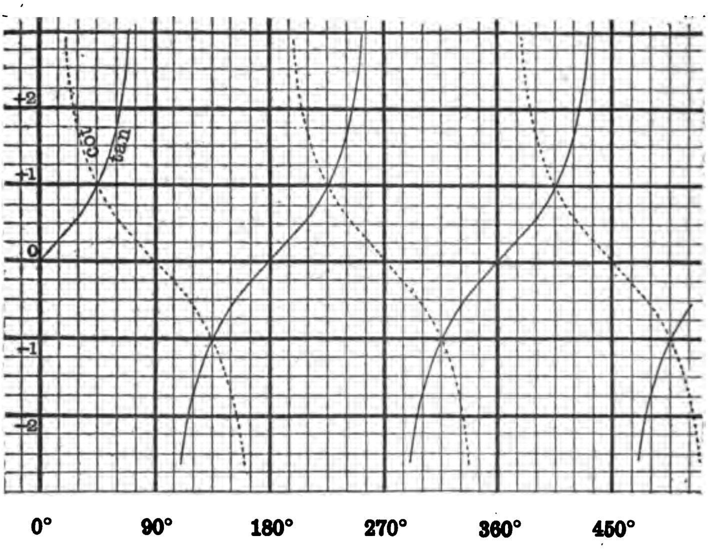
$$f(x) = f(x \pm n \cdot 360°),$$

where f stands for any one of the trigonometric functions.

Hence the trigonometric functions are *periodic,* with a *period of* 360°. (See graphs below.)

15. Graphs of the Trigonometric Functions.





16. Relations between the Functions of an Angle. — From the general definitions of the functions given in (**10**), putting $\angle XOP = x$, we find that

$$\sin x = \frac{1}{\csc x}; \quad \cos x = \frac{1}{\sec x}; \quad \tan x = \frac{1}{\cot x}.$$

$$\tan x = \frac{\text{ordinate}}{\text{abscissa}} = \frac{\dfrac{\text{ordinate}}{\text{distance}}}{\dfrac{\text{abscissa}}{\text{distance}}} = \frac{\sin x}{\cos x}; \quad \cot x = \frac{\cos x}{\sin x}.$$

Also, whatever be the quadrant of angle $XOP = x$ [figure of (**10**)], we have

$$(\text{ordinate})^2 + (\text{abscissa})^2 = (\text{distance})^2.$$

Dividing this equation through in turn by (distance)^2, (abscissa)^2, and (ordinate)^2, and expressing the resulting ratios as functions we have

$$\sin^2 x + \cos^2 x = 1,$$
$$1 + \tan^2 x = \sec^2 x,$$
$$1 + \cot^2 x = \csc^2 x.$$

All the above relations between the functions of an angle x are true for all values of x. They form a first set of working formulas, and should be thoroughly committed to memory. They are collected below, as

Formulas, Group A

(1) $\sin x = \dfrac{1}{\csc x}$.

(2) $\cos x = \dfrac{1}{\sec x}$.

(3) $\tan x = \dfrac{1}{\cot x}$.

(4) $\tan x = \dfrac{\sin x}{\cos x}$.

(5) $\cot x = \dfrac{\cos x}{\sin x}$.

(6) $\sin^2 x + \cos^2 x = 1$.

(7) $1 + \tan^2 x = \sec^2 x$.

(8) $1 + \cot^2 x = \csc^2 x$.

We shall apply these formulas in two examples.

Example 1. Prove that $\tan x + \cot x = \sec x \csc x$.

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}$$
$$= \frac{1}{\sin x \cos x} = \frac{1}{\sin x} \cdot \frac{1}{\cos x} = \csc x \sec x.$$

Example 2. Prove that

$$\frac{\csc x}{\tan x + \cot x} = \cos x.$$

$$\frac{\csc x}{\tan x + \cot x} = \frac{\csc x}{\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}} = \frac{\csc x}{\dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x}}$$

$$= \frac{\csc x}{\dfrac{1}{\sin x \cos x}} = \csc x \sin x \cos x = \cos x.$$

In both examples all the steps taken are true for all values of x, since this is true of all the formulas of group A. Hence the given equations are true for all values of x, and they are therefore called *trigonometric identities.*

The equation $\sin^2 x - \cos^2 x = 1$ is not true for all values of x, but holds only for certain special values; it is not an identity.

Exercises. Prove the following identities:

1. $\sin x \cot x = \cos x.$

2. $\dfrac{1}{\tan x \csc x} = \cos x.$

3. $\dfrac{\sec x}{\tan x} = \csc x.$

4. $\dfrac{\csc x}{\cot x} = \sec x.$

5. $(\sin^2 x + \cos^2 x)^2 = 1.$

6. $\dfrac{\sin\theta}{\cos\theta \cot\theta} = \tan^2\theta.$

(For names of Greek letters see first page of appendix.)

7. $(\csc\theta - \cot\theta)(\csc\theta + \cot\theta) = 1.$

8. $(\sec x - \tan x)(\sec x + \tan x) = 1.$

9. $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta.$

10. $\sin^2\alpha + \cos^2\alpha = \csc^2\alpha - \cot^2\alpha.$

11. $(\sin\alpha - \cos\alpha)^2 = 1 - 2\sin\alpha\cos\alpha.$

12. $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x.$

13. $(1 - \cos^2 x)\sec^2 x = \tan^2 x.$

14. $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta.$

15. $\sec\theta\csc\theta - \cot\theta = \tan\theta.$

16. $\cot\phi\cos\phi + \sin\phi = \csc\phi.$

17. $\cos^2\phi\csc^2\phi = \csc^2\phi - 1.$

18. $\dfrac{\sin\phi}{1 + \cos\phi} = \dfrac{1 - \cos\phi}{\sin\phi}.$

19. $\dfrac{1 + \tan^2\beta}{1 + \cot^2\beta} = \dfrac{\sin^2\beta}{\cos^2\beta}.$

20. $(1 - \cos^2\beta)(1 + \cot^2\beta) = 1.$

21. $\tan^4 x - \sec^4 x = 1 - 2\sec^2 x.$

22. $\dfrac{\cos x + \sin x}{\cos x - \sin x} = \dfrac{1 + \tan x}{1 - \tan x}.$

23. $(\tan x - 1)(\cot x - 1) = 2 - \sec x \csc x.$

24. $\csc\theta + \cot\theta = \dfrac{\sin\theta}{1 - \cos\theta}.$

25. $(a\cos x - b\sin x)^2 + (a\sin x + b\cos x)^2 = a^2 + b^2.$

26. $\cos^2\phi + (\sin\phi\cos\theta)^2 + (\sin\phi\sin\theta)^2 = 1.$

27. $\tan\alpha + \tan\beta = \tan\alpha\tan\beta(\cot\alpha + \cot\beta).$

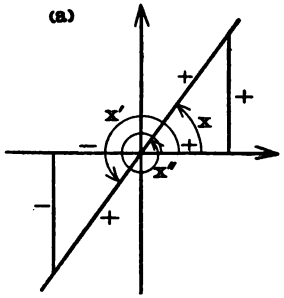
17. Functions of any Angle in Terms of Functions of an Acute Angle. — It is possible to express in a simple manner any function of any angle in terms of a function of an acute angle. Therefore a table of values of the functions of angles from 0° to 90° will serve for all angles. In fact, in view of (6), a table of functions from 0° to 45° would be sufficient, though not convenient.

1. Any angle, positive or negative, can be brought into the first quadrant by adding to it, or subtracting from it, an integral multiple of 90°.

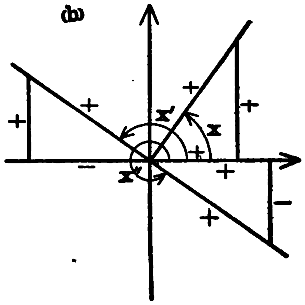
Thus: $760° - 8 \times 90° = 40°; \quad -470° + 6 \times 90° = 70°.$

2. When an angle is changed by an integral multiple of 90°, say $n \times 90°$, the new terminal line lies in the *same line* as the original terminal line when *n is even; at right angles* to it when *n is odd.*

3. Two angles which differ by an *even multiple of* 90° will be called *symmetrical* with respect to the initial line, or simply *symmetrical;* two angles which differ by an *odd multiple of* 90°, *skew-symmetrical.*



Angles x' and x'' are symmetrical with respect to angle x



Angles x' and x'' are skew-symmetrical with respect to x

4. When two angles are *symmetrical,* any function of the one is *numerically* equal to the *same function* of the other.

From figure (a), $\sin x = -\sin x' = \sin x''$, etc., for the other functions.

When two angles are *skew-symmetrical,* any function of the one is *numerically* equal to the *co-function* of the other.

From figure (b), $\sin x = -\cos x' = \cos x''$, etc., for the other functions.

Exercise 1. From figures (a) and (b), write down all the functions of x in terms of functions of x' and of x''.

Exercise 2. Draw figures corresponding to figures (a) and (b), when x lies in each of the other quadrants. Then proceed as in exercise 1.

5. **Rule:** *Any function of any angle x is numerically equal to the* $\left\{\begin{matrix}\textit{same function}\\ \textit{co-function}\end{matrix}\right.$ *of x increased or diminished by any* $\left\{\begin{matrix}\textit{even}\\ \textit{odd}\end{matrix}\right.$ *multiple of* 90°.

The sign of the result must be determined by noting the quadrants of x and $x \pm n \cdot 90°$.

When the new angle, $x \pm n \cdot 90°$, lies in the first quadrant, give to the result the sign of the given function of x.

Examples. (Let the student draw the necessary figures.)

1. $\sin 680° = \sin(50° + 7 \times 90°) = -\cos 50°$.

Here we diminish the given angle by an odd multiple of 90°, hence change to the co-function. Also sin 680° is negative, hence we use the minus sign.

2. $\tan(-870°) = \tan(30° - 10 \times 90°) = +\tan 30°$.

3. $\sec 680° = \sec(230° + 5 \times 90°) = -\csc 230°$.

Here we use the negative sign because, by noting the quadrants of the angles, we see that sec 680° and csc 230° must be of opposite signs.

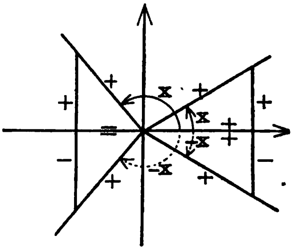
Exercises.

1. Express the functions of 520° in terms of those of 70°; of those of 160°; of 250°; of 340°.

2. Express the functions of − 230° in terms of those of 40°; of 220°; of 400°.

18. Relations between the Functions of $+x$ and $-x$.—The figure shows two cases, x in the first quadrant and x in the second quadrant. In either case,

$$\begin{aligned}
\sin x &= -\sin(-x);\\
\csc x &= -\csc(-x);\\
\cos x &= \cos(-x);\\
\sec x &= \sec(-x);\\
\tan x &= -\tan(-x);\\
\cot x &= -\cot(-x).
\end{aligned}$$



Exercise. Show that these equations are true when x lies in the third quadrant or fourth quadrant.

Rule: *The cosine or secant of any angle is equal to the cosine or secant respectively of the negative angle; the remaining four functions of the angle are equal to the negative of the corresponding functions of the negative angle.* Or,

$f(x) = f(-x)$ *when f stands for cos. or sec.*
$f(x) = -f(-x)$ *when f stands for sin., csc., tan., or cot.*

19. Exercises. Express all the functions of the following angles in terms of functions of acute angles:

1. 140°.
2. 155°.
3. 235°.
4. 335°.
5. 355°.
6. – 35°.
7. – 115°.
8. – 255°.
9. – 318°.
10. 738°.
11. – 670°.
12. 1120°.
13. – 1040°.
14. – 410°.
15. 535°.
16. – 103°.

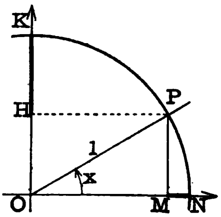
Express all the functions of the following angles in terms of functions of angles between 0° and 45°.

17. 75°.
18. – 80°.
19. 110°.
20. 255°.
21. – 335°.
22. 600°.
23. 790°.
24. – 510°.

Give the exact values of the functions of:

25. 120°.
26. 135°.
27. 150°.
28. 300°.
29. – 30°.
30. – 45°.
31. – 60°.
32. – 120°.
33. – 240°.
34. 315°.
35. 600°.
36. – 510°.

20. Versed Sine and Coversed Sine. — The expressions $1 - \cos x$ and $1 - \sin x$ occur often enough in the applications of trigonometry to warrant the use of special symbols for them. These are



$$\mathbf{1 - \cos x \equiv \text{versed sine of } x \equiv \text{vers } x;}$$
$$\mathbf{1 - \sin x \equiv \text{coversed sine of } x \equiv \text{covers } x.}$$

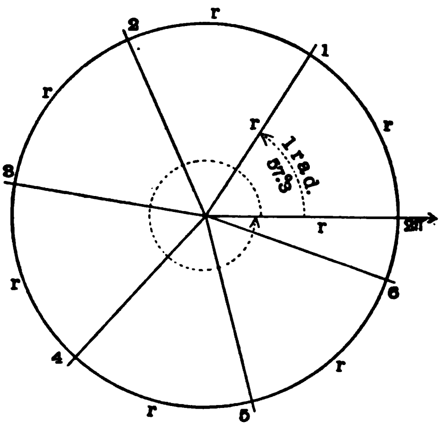
Their line values are (figure), vers $x = MN$, covers $x = HK$, x being in the first quadrant.

Exercises. Find the values of the versed sine and coversed sine of:

1. 30°.
2. 45°.
3. 60°.
4. 90°.
5. 120°.
6. 135°.
7. 150°.
8. – 30°.
9. – 120°.
10. – 225°.
11. – 300°.
12. – 315°.

21. Radian Measure. — The *degree* is an artificial unit for the measurement of angles. In France, where at the time of the Revolution an attempt was made to put all measurements on the basis of the decimal scale, the quadrant of the circle was divided into 100 equal parts and the angle subtended at the center by one part called a *grade*. Each grade was then subdivided into 100 equal parts called *minutes*, and each minute into 100 *seconds*. The degree and the grade are thus two arbitrary units for the measurement of angles, and any number of such units might be chosen.

There is one unit which is naturally related to the circle, and which is as commonly used in theory as the degree in practice. *It is the central angle subtended by an arc equal in length to the radius of the circle, and is called a* ***radian*** (figure).



Since the circumference contains the radius 2π times, the entire central angle of 360° contains 2π radians, i.e.,

$$\mathbf{2\pi \text{ radians} = 360°.}$$

Hence,

π radians = 180°;

$\frac{\pi}{2}$ radians = 90°;

$\frac{\pi}{4}$ radians = 45°; and so on.

In dealing with angles measured in radians it is customary to omit specifying the unit used; it is understood that when no unit is indicated the radian is implied. Thus,

$$2\pi = 360°, \quad \pi = 180°, \quad \frac{\pi}{3} = 60°, \quad 2\tfrac{1}{2} = 2\tfrac{1}{2} \text{ radians, and so on.}$$

NOTE. To get the standard form of the graphs of the equations $y = \sin x$, $y = \cos x$, etc., take x in radians on the x-axis, thus: $x = 0.1, 0.2, 0.3, \ldots, 1, \ldots$ and find the corresponding values of y; use the same unit of length for both x and y.

22. Radians into degrees, and conversely.

Since 2π (radians) = 360°,

therefore, $$\mathbf{1\ radian} = \frac{360°}{2\pi} = \frac{180°}{\pi} = \frac{180°}{3.1416-} = \mathbf{57°.29+};$$

also, $$\mathbf{1\ degree} = \frac{2\pi}{360}\text{(radians)} = \frac{\pi}{180}\text{ (radians)}$$

$$= \frac{1}{57.29+}\text{(radians)} = \mathbf{.017+}\text{(radians)}.$$

Rule: *To convert radians into degrees, multiply the number of radians by* $\frac{180}{\pi}$ *or* 57.29+.

To convert degrees into radians, multiply the number of degrees by $\frac{\pi}{180}$ *or* $\frac{1}{57.29+}$ *or* .017+.

By taking a sufficiently accurate value of π, we find,

1 radian $= 57^\circ.2957795 = 3437'.74677 = 206264''.8$.
$1^\circ = .0174533$ radians.
$1' = .0002909$ radians (point, 3 ciphers, 3, approx.).
$1'' = .0000048$ radians (point, 5 ciphers, 5, approx.).

The measure of an angle in radians is often called the *circular measure* of the angle.

Examples.

1. Reduce 240° to radians.

$$240^\circ = 240 \times \frac{\pi}{180} \text{ radians} = \frac{4\pi}{3} \text{ radians.}$$

2. Reduce to degrees the angle whose radian measure is $1 + \pi$.

$$(1+\pi) \text{ radians} = (1+\pi) \times \frac{180}{\pi} \text{ degrees} = \left(\frac{180}{\pi} + 180\right) \text{degrees}$$
$$= 57^\circ.29 + 180^\circ \qquad = 237^\circ.29+.$$

3. Reduce to degrees the angle whose circular measure is $\dfrac{2}{\pi - 1}$ radians.

We can see that, since $\pi = 3.14+$, the given fraction has a value a little less than 1; hence the angle is a little less than one radian, hence less than 57°.3. Making the reduction we have

$$\begin{aligned}\frac{2}{\pi-1} \text{ radians} &= \frac{2}{\pi-1} \times \frac{180}{\pi} \text{ degrees}\\ &= \frac{360}{\pi^2 - \pi} \text{ degrees}\\ &= \frac{360}{6.73\pm} \text{ degrees}\\ &= 53^\circ.5\pm.\end{aligned}$$

See also Table IV, Appendix.

23. Exercises. Reduce to degrees, minutes and seconds the angles whose circular measures are:

1. $\dfrac{\pi}{12}, \dfrac{5\pi}{3}, \dfrac{5\pi}{16}, \dfrac{8\pi}{5}, \dfrac{22\pi}{15}$.

2. $2, 1.5, \dfrac{3}{2}, \dfrac{4}{3}, \dfrac{8}{5}$.

3. $\dfrac{5\pi}{12}, -\dfrac{3}{2}, \dfrac{2\pi}{15} + 1, \dfrac{\pi}{3} - \dfrac{2}{3}, \dfrac{2\pi + 5}{8}$.

4. $\dfrac{1}{4} + \pi, \dfrac{\pi}{4} - \dfrac{1}{3}, \dfrac{1}{\pi}, \dfrac{2}{\pi - 3}, \pi^2$.

5. $\dfrac{\pi}{\pi^2 + 1}, \dfrac{\pi^2}{1 - \pi}, \dfrac{\pi + 1}{\pi - 1}$.

Reduce the following angles to circular measure:

6. 30°, 120°, 150°, 225°, −60°.

7. 375°, $-22\frac{1}{2}$°, 187°.5, 106°, 93° 45′.

8. 85°, 191° 15′, 5° 37′ 30″, 90° 37′ 30″.

9. 10′, 10″, 0″.1, 12° 5′ 4″, 21° 36′ 8″.1.

10. If the radius of the earth be taken as 3960 miles, find the number of feet in an arc of 1″ of the meridian.

11. How many radians in a central angle subtended by an arc 150 ft. long, the radius of the circle being 50 ft.?

12. How many radians in the central angle subtended by the side of a regular inscribed decagon?

13. A wheel makes 1000 revolutions a minute. Find its angular velocity in radians per second.

14. If the angular velocity of a wheel is 3π radians per second, how many revolutions per minute does it make?

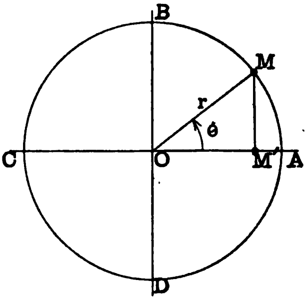
15. Show that a ball, viewed from a distance equal to 57 times its diameter, will subtend at the eye an angle of nearly 1°; at a distance of 3400 times the diameter the angle will be very nearly 1′; at a distance of 206,000 times its diameter the angle will be almost exactly 1″.

16. At what distance from the eye will a baseball subtend an angle of 1°? Of 1′? Of 1″?

17. The moon's diameter is 2160 miles, the sun's 866,000 miles. Their distances from the earth are 240,000 miles and 93,000,000 miles respectively. What is the angular diameter of each body as viewed from the earth?

18. Is the end of a lead pencil, held at arm's length, sufficient to cover the disk of the full moon? Moon's angular diameter is 32′.

24. Use of Radian Measure. — Suppose a particle M to be moving with uniform speed of 20 ft. per second around a circle of radius 5 ft.



Suppose the particle to start from A and let t (time) be the number of seconds required to reach the position M; let $\angle AOM = \theta$. (We may imagine the radius OM to be revolving with the particle M, and the angle θ to be increasing gradually.)

Now in t seconds after the particle has left the point A, arc AM will have the length $20\,t$ ft. Therefore

$$\textit{radian measure of } \angle\theta = \frac{\textit{arc } AM}{\textit{radius}} = \frac{20\,t}{5} = 4\,t.$$

That is, *in t seconds the angle AOM reaches the value* 4 *t radians.*

Let M' be the projection of M on OA. Then the distance of M' from O at any time t is

$$OM' = OM \cos \theta = 5 \cos 4\,t.$$

This will make OM' come out negative when M' lies to the left of O. Thus:

When $t = 0.2$ sec; $OM' = 5 \cos \frac{4}{5}$ (radians) $= 5 \cos 45°.8 = 3.48$ ft.
When $t = 1$ sec; $OM' = 5 \cos 4$ (radians) $= 5 \cos 229°.2 = -3.27$ ft.

Exercise 1. Calculate OM' when $t = \frac{1}{8}, \frac{1}{4}, \frac{2}{3}, \frac{4}{3}, 2, 5$ seconds.

Exercise 2. If the speed of M is v ft. per second and $OM = r$ ft., show that $\theta = \frac{v}{r}t$ radians, and $OM' = r \cos \frac{v}{r}t$.

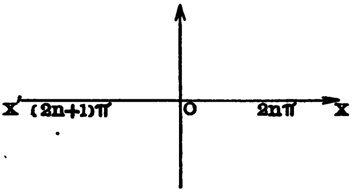
Angular speed. The angle through which radius OM turns per second is called the *average angular speed* of M. It is the angle turned through in t seconds divided by the number of seconds. Using the result of Exercise 2 above, we have

$$\textit{Average angular speed} = \frac{\theta\ (\textit{radians})}{t} = \frac{v}{r}t \div t = \frac{v}{r}\ \textit{radians}.$$

What is the angular speed in the above illustration?

Exercise. What is the angular speed of a rotating wheel of radius 10 inches, if a point on the rim has a speed of 25 inches per second? Give the answer in radians and also in degrees.

25. Angles Corresponding to a Given Function. — Let n denote an integer, positive or negative, or zero; then $2\,n$ is always even, and $2\,n + 1$ odd; hence the angle $2\,n\pi$ has the terminal line OX (figure) coincident with the initial line, and angle $(2\,n + 1)\,\pi$ has the terminal line OX'.

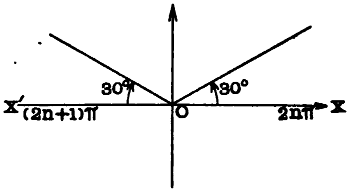


Suppose now we wish to write down all angles x such that $\sin x = \frac{1}{2}$. Corresponding to a given function, there are always (except when the angle is a multiple of 90°) two angles less than 360°; in this case they are

$$30° \text{ and } \pi - 30°.$$

All angles with the same terminal line as either one of these will have the same functions; all such angles are

$$2n\pi + 30^\circ \text{ and } 2n\pi + (\pi - 30^\circ) = (2n+1)\pi - 30^\circ.$$



Hence all solutions of the equation $\sin x = \frac{1}{2}$ are given by

$$x = 2n\pi + 30^\circ \text{ or } (2n+1)\pi - 30^\circ.$$

In general, if θ denote the smallest positive angle whose sine is a, then all solutions of the equation

(1) $\sin x = a$ are $x = 2n\pi + \theta$ and $(2n+1)\pi - \theta$.

Hence also, if θ denote the smallest positive angle whose cosecant is a, the solutions of the equation

(2) $\csc x = a$ are $x = 2n\pi + \theta$ and $(2n+1)\pi - \theta$.

Consider next the equation

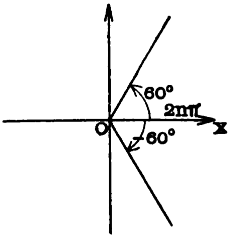
$$\cos x = \tfrac{1}{2}.$$

The two simplest solutions are

$$x = +60^\circ \text{ and } x = -60^\circ.$$

All possible solutions are given by

$$x = 2n\pi + 60^\circ \text{ and } x = 2n\pi - 60^\circ,$$
$$\text{or } x = 2n\pi \pm 60^\circ.$$



In general, if θ be the smallest positive angle whose cosine is a, all solutions of the equation

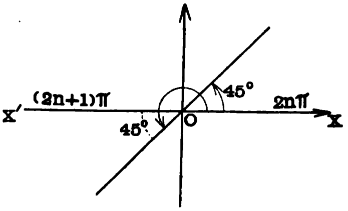
(3) $\cos x = a$ are $x = 2n\pi \pm \theta$.

Hence also, if θ be the smallest angle whose secant is a, all solutions of the equation

(4) $\sec x = a$ are $x = 2n\pi \pm \theta$.

Finally consider the equation

$$\tan x = 1.$$



The two simplest solutions are

$$x = 45° \quad \text{and} \quad x = \pi + 45°,$$

and all possible solutions are

$$x = 2\,n\pi + 45°$$

and $$x = 2\,n\pi + (\pi + 45°),$$

the second set being the same as $x = (2\,n + 1)\,\pi + 45°$.

Both sets are contained in the single equation

$$x = n\pi + 45°,$$

the first set being obtained when n is even, the second set when n is odd.

In general, if θ be the smallest positive angle whose tangent is a, all solutions of the equation

(5) $\tan x = a$ are $x = n\pi + \theta.$

Hence also, if θ be the smallest positive angle whose cotangent is a, all solutions of the equation

(6) $\cot x = a$ are $x = n\pi + \theta.$

Summary of equations (1) to (6).

Let θ denote the smallest positive angle having a given function equal to a given number a. *Then all solutions of the equation*

$$\text{I.}\quad \left\{\begin{array}{l}\sin x = a\\ \csc x = a\end{array}\right. \quad \text{are} \quad x = 2\,n\pi + \theta \quad \text{and} \quad (2\,n + 1)\,\pi - \theta;$$

$$\text{II.}\quad \left\{\begin{array}{l}\cos x = a\\ \sec x = a\end{array}\right. \quad \text{are} \quad x = 2\,n\pi \pm \theta;$$

$$\text{III.}\quad \left\{\begin{array}{l}\tan x = a\\ \cot x = a\end{array}\right. \quad \text{are} \quad x = n\pi + \theta.$$

The angle θ is usually called the **principal value** of x.

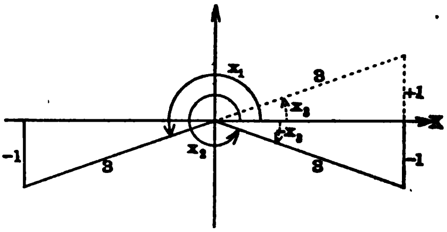
The preceding discussion leads to the following simple rule.

Rule: *Corresponding to a given value of a function, there are in general two and only two positive angles less than* 360°. *If these be denoted by* x_1 *and* x_2, *then all possible angles are given by* $x_1 \pm 2\,n\pi$ *and* $x_2 \pm 2\,n\pi$.

In exceptional cases there may be only one angle $< 360°$, as when $\sin x = 1$ or $\cos x = -1$.

26. Use of Tables of Natural Functions. — Such tables give the functions of angles from 0° to 90°. But they will serve for all four quadrants, since any function of any angle is reducible to a function of an acute angle.

When the given value of the function is not found exactly in the table, the corresponding angle must be obtained by *interpolation*.



Example 1. Given $\sin x = -\frac{1}{3}$. To find x.

The two values, x_1 and x_2, $< 360°$, are shown in the figure. They are easily found when x_3, the angle whose sine is $+\frac{1}{3}$, is known. For

$$x_1 = \pi + x_3 \quad \text{and} \quad x_2 = 2\pi - x_3.$$

Since $\sin x_3 = \frac{1}{3} = 0.333$, we find by interpolation from Table III, $x_3 = 19° 28'$. Hence,

$$x_1 = 199° 28', \quad x_2 = 340° 32'.$$

All possible values of x are then given by

$$199° 28' \pm 2n\pi, \quad 340° 32' \pm 2n\pi.$$

Example 2. Given $\cot \frac{2}{3}x = 3.362$. To find x.

From Table III, $\frac{2}{3}x = 16° 34'$ or $196° 34'$ ($= 180° + 16° 34'$).

Hence all possible values of $\frac{2}{3}x$ are given by

$$\tfrac{2}{3}x = 16° 34' \pm 2n\pi \quad \text{or} \quad 196° 34' \pm 2n\pi.$$

Therefore, $x = 24° 51' \pm 3n\pi$ or $294° 51' \pm 3n\pi$.

We might also write, from III of (**25**),

$$\tfrac{2}{3}x = 16° 34' + n\pi; \text{ hence } x = 24° 51' + \tfrac{3}{2}n\pi.$$

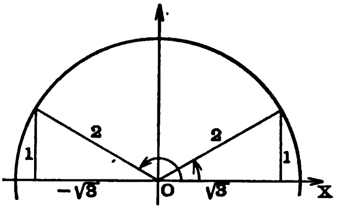
27. Exercises. Find all values of the angles which satisfy the following equations:

1. $\cot x = 1$; $\sin x = -\frac{1}{2}$; $\sec x = 2$; $\cos x = 1$.
2. $\csc x = -\sqrt{2}$; $\tan x = \sqrt{3}$; $\cos x = .5$; $\cot x = -\sqrt{3}$.
3. $\sin x = -\frac{3}{4}$; $\sec x = -4$; $\tan x = 3$; $\csc x = 3$.
4. $\cos x = -.275$; $\cot x = -.948$; $\sin x = .035$.
5. $\tan\theta = 1.555$; $\csc\theta = -3.675$; $\sec\theta = -5$.
6. $\text{vers}\,\phi = 2.560$; $\text{vers}\,\phi = .712$; $\text{covers}\,\phi = .813$.

28. Given one function of an angle, to find the other functions.

Example 1. $\sin x = \frac{1}{2}$. Find the other functions.

Take ordinate = 1 and distance = 2; then abscissa = $\pm\sqrt{3}$ (figure).



Then

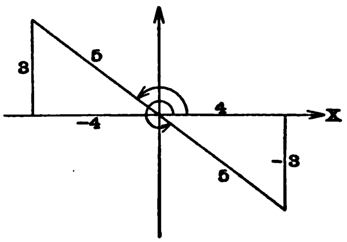
$$\cos x = \pm\frac{\sqrt{3}}{2},\quad \tan x = \pm\frac{1}{\sqrt{3}},$$

$$\cot x = \pm\sqrt{3},\quad \sec x = \pm\frac{2}{\sqrt{3}},\quad \csc x = 2.$$

We have found *two values* for each function except $\csc x$, which is the reciprocal of the given function. Similar results will be found in general.

Example 2.

$$\tan x = -\frac{3}{4}\ \left(= \frac{-3}{+4} \text{ or } \frac{+3}{-4}\right).$$



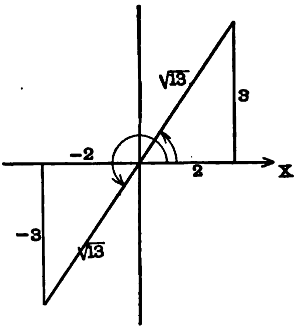
The two possible positions of the terminal line are shown in the figure.

Hence, $\sin x = \pm\frac{3}{5}$, $\cos x = \mp\frac{4}{5}$,

$$\cot x = -\frac{4}{3},\ \csc x = \pm\frac{5}{3},\ \sec x = \mp\frac{5}{4}.$$

Example 3.

$$\cot x = \frac{2}{3}\ \left(= \frac{+2}{+3} \text{ or } \frac{-2}{-3}\right)$$



Then (figure),

$$\sin x = \pm\frac{3}{\sqrt{13}},\quad \cos x = \pm\frac{2}{\sqrt{13}},$$

$$\tan x = \frac{3}{2},$$

$$\csc x = \pm\frac{\sqrt{13}}{3},\quad \sec x = \pm\frac{\sqrt{13}}{2}.$$

Example 4. $\sin x = \frac{h}{k}$.

Ordinate = h; distance = k; hence abscissa = $\pm\sqrt{k^2 - h^2}$.

Then $$\cos x = \pm\frac{\sqrt{k^2 - h^2}}{k},\quad \tan x = \pm\frac{h}{\sqrt{k^2 - h^2}},\ \text{etc.}$$

Exercise 1. Construct figures for the cases when $\frac{h}{k}$ is (a) plus; (b) minus.

Exercise 2. Is the problem possible for all values of h and k?

Exercise 3. If $\tan x = \frac{a-b}{2\sqrt{ab}}$ show that

$$\sin x = \pm \frac{a-b}{|a+b|}, \quad \cos x = \pm \frac{2\sqrt{ab}}{|a+b|}, \text{ etc.}$$

Calculate the values of the six functions when $a = 2$, $b = 3$; when $a = -2$, $b = -3$; when $a = 1$, $b = 4$; $a = -1$, $b = -4$.

Exercise 4. Is Exercise 3 possible for all values of a and b?

29. Exercises. Find the other functions, given that

1. $\sin x = -\frac{2}{3}$.
2. $\cos x = \frac{2}{3}$.
3. $\tan x = -\frac{3}{4}$.
4. $\sec x = 5$.
5. $\cot x = -\sqrt{3}$.
6. $\csc x = -\frac{13}{5}$.
7. $\sec x = -\frac{41}{9}$.
8. $\cot x = -.8$.
9. $\sin x = -.8$.
10. $\cos \theta = \sqrt{a}$.
11. $\csc \theta = -m$.
12. $\tan \theta = \frac{a}{b}$.
13. $\sin \phi = 1 + h$.
14. $\cot \phi = \sqrt{a-1}$.
15. $\sec \phi = \frac{2ab}{a^2+b^2}$.

16. State for what values of the literal quantities in exercises 10–15, the given equations are impossible.

30. To express all the functions in terms of one of them.

1. Express all the functions in terms of the cosine.

We have

$$\cos x = \frac{\cos x}{1} = \frac{\text{abscissa}}{\text{distance}}.$$

Hence let abscissa = $\cos x$ and distance = 1.

Then ordinate $= \pm \sqrt{\text{dist.}^2 - \text{absc.}^2} = \pm \sqrt{1 - \cos^2 x}$.

The figure shows this graphically when $\cos x$ is positive.

Taking into account both values of the ordinate, we have

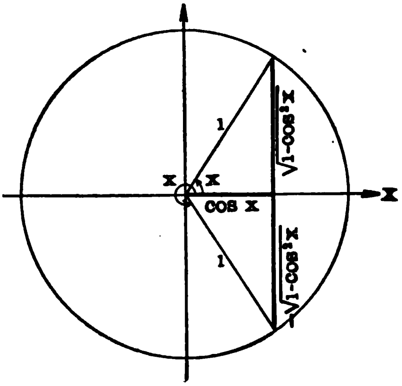
$$\sin x = \pm \sqrt{1-\cos^2 x};$$

$$\tan x = \pm \frac{\sqrt{1-\cos^2 x}}{\cos x};$$

$$\cot x = \pm \frac{\cos x}{\sqrt{1-\cos^2 x}};$$

$$\csc x = \pm \frac{1}{\sqrt{1-\cos^2 x}};$$

$$\sec x = \frac{1}{\cos x}.$$



Exercise 1. Draw a figure for the case when $\cos x$ is negative.

Exercise 2. Obtain the same equations directly from the formulas of Group A.

2. Express all the functions in terms of the cotangent.

$$\cot x = \frac{\cot x}{1} = \frac{-\cot x}{-1} = \frac{\text{abscissa}}{\text{ordinate}}.$$

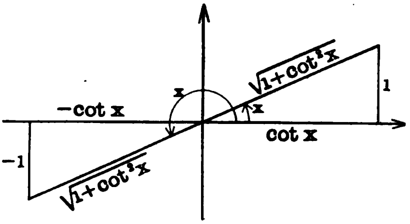
Hence let abscissa $= \cot x$ and ordinate $= 1$.
or let abscissa $= -\cot x$ and ordinate $= -1$.

In either case, distance $= + \sqrt{1 + \cot^2 x}$. (See figure, where we assume $\cot x > 0$.)

$$\text{Hence}\quad \sin x = \pm \frac{1}{\sqrt{1 + \cot^2 x}},\quad \cos x = \pm \frac{\cot x}{\sqrt{1 + \cot^2 x}},\ \text{etc.}$$

Exercises.

1. By taking each of the functions in turn, and proceeding as above, obtain the results shown in the following table. The given function and its reciprocal are uniquely determined; the other four functions are ambiguous in sign.



	sin x.	cos x.	tan x.	cot x.	sec x.	csc x.
sin x	....	$\pm\sqrt{1-\cos^2 x}$	$\frac{\tan x}{\pm\sqrt{1+\tan^2 x}}$	$\frac{1}{\pm\sqrt{1+\cot^2 x}}$	$\frac{\pm\sqrt{\sec^2 x-1}}{\sec x}$	$\frac{1}{\csc x}$
cos x	$\pm\sqrt{1-\sin^2 x}$	....	$\frac{1}{\pm\sqrt{1+\tan^2 x}}$	$\frac{\cot x}{\pm\sqrt{1+\cot^2 x}}$	$\frac{1}{\sec x}$	$\frac{\pm\sqrt{\csc^2 x-1}}{\csc x}$
tan x	$\frac{\sin x}{\pm\sqrt{1-\sin^2 x}}$	$\frac{\pm\sqrt{1-\cos^2 x}}{\cos x}$	....	$\frac{1}{\cot x}$	$\pm\sqrt{\sec^2 x-1}$	$\frac{1}{\pm\sqrt{\csc^2 x-1}}$
cot x	$\frac{\pm\sqrt{1-\sin^2 x}}{\sin x}$	$\frac{\cos x}{\pm\sqrt{1-\cos^2 x}}$	$\frac{1}{\tan x}$	....	$\frac{1}{\pm\sqrt{\sec^2 x-1}}$	$\pm\sqrt{\csc^2 x-1}$
sec x	$\frac{1}{\pm\sqrt{1-\sin^2 x}}$	$\frac{1}{\cos x}$	$\pm\sqrt{1+\tan^2 x}$	$\frac{\pm\sqrt{1+\cot^2 x}}{\cot x}$	....	$\frac{\csc x}{\pm\sqrt{\csc^2 x-1}}$
csc x	$\frac{1}{\sin x}$	$\frac{1}{\pm\sqrt{1-\cos^2 x}}$	$\frac{\pm\sqrt{1+\tan^2 x}}{\tan x}$	$\pm\sqrt{1+\cot^2 x}$	$\frac{\sec x}{\pm\sqrt{\sec^2 x-1}}$	....

2. Express $\cos x \sin^2 x + \cos^3 x$ in terms of $\tan x$.

3. Express $\cot x \csc x + \csc^2 x$ in terms of $\sin x$.

4. Express $\cos^2 x \cot x - \sin^2 x \tan x$ in terms of $\cot x$.

5. Express $\dfrac{1}{1+\sin\theta} + \dfrac{1}{1-\sin\theta}$ in terms of $\csc\theta$.

6. Express $\dfrac{\cos\theta}{1-\tan\theta} + \dfrac{\sin\theta}{1-\cot\theta}$ in terms of $\sec\theta$.

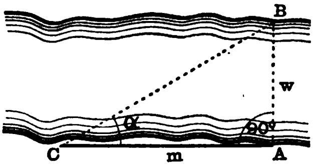
Note. Before proceeding to the next chapter the student should become familiar with the theory and use of logarithms. These are explained at the end of the book, preceding the tables.

CHAPTER II

SOME APPLICATIONS OF THE TRIGONOMETRIC FUNCTIONS

Part I. Heights and Distances

31. Width of a River. — To determine the width of a river, $w = AB$, a surveyor might set his transit at A, sight across to a well marked point B, turn off 90° into the line AC, and have a stake set at some convenient point C. Measure $AC = m$, and from C measure $\angle ACB = \alpha$.



Then from $\triangle ABC$ we have

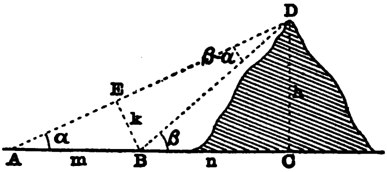
$$\frac{w}{m} = \tan \alpha, \quad \text{or,} \quad w = m \tan \alpha.$$

Exercise. Calculate w when

	(*a*)	(*b*)	(*c*)	(*d*)
$m =$	227 ft.	129.5 ft.	663 ft.	387 ft.
$\alpha =$	51° 43′.	31° 26′.	42° 17′.	19° 33′.

Note. Logarithms should be used in these calculations. Check results by measurement of figures drawn to scale.

32. Height of an Inaccessible Object. — To find h, the height of a hill, say, choose a point A on level ground and measure $\angle CAD = \alpha$, called "the angle of elevation." Then approach a measured distance m on level ground, to B; at B measure the angle of elevation β. Now α, β, and m are known; to calculate h.



First Solution. Let $BC = n$.

Then $$\frac{n}{h} = \cot \beta, \quad \text{and} \quad \frac{m+n}{h} = \cot \alpha.$$

Subtracting: $$\frac{m}{h} = \cot \alpha - \cot \beta; \quad \text{hence} \quad h = \frac{m}{\cot \alpha - \cot \beta}.$$

Second Solution. Let k be the length of the perpendicular from B on AD. Then we can calculate, in order, first k, next BD, and finally h.

From $\triangle ABE$: $\quad k = m \sin \alpha.$

From $\triangle BED$: $\quad BD = \dfrac{k}{\sin(\beta - \alpha)} = m \dfrac{\sin \alpha}{\sin(\beta - \alpha)}.$

From $\triangle BDC$: $\quad h = BD \sin \beta = m \dfrac{\sin \alpha \sin \beta}{\sin(\beta - \alpha)}.$

For logarithmic calculation this formula is much better than the preceding. It gives

$$\log h = \log m + \log \sin \alpha + \log \sin \beta + \operatorname{colog} \sin(\beta - \alpha).$$

Exercises.

1. What does the second solution give when $\beta = 2\alpha$? Explain.
2. Use both formulas to find h when

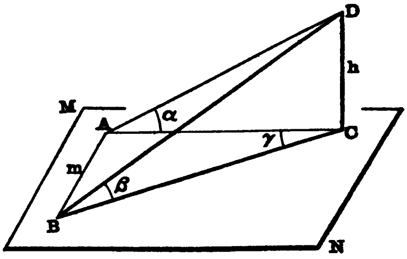
	(*a*)	(*b*)	(*c*)
$\alpha =$	20°,	15° 48′,	27° 33′,
$\beta =$	25°,	22° 17′,	41° 07′,
$m =$	350 ft.	189.7 ft.	228.3 ft.

Draw figures to scale and give the graphic solutions.

3. From a point on the bank of a river the angle of elevation of the top of a tree on the opposite bank is 38° 52′; from a point 200 ft. straight back from the bank the angle of elevation is 19° 26′. Find the height of the tree and the width of the river. Also give graphic solution.

33. Height of an Inaccessible Object. Second Method. —

Let CD stand perpendicular to the horizontal plane MN. To determine the height CD or h.



From A measure $\angle \alpha$; if now we cannot approach C or recede from it on account of obstacles such as trees, or a river, or other barrier, lay off a measured distance $AB = m$, at right angles to AC; at B measure $\angle \beta$.

Given m, α, β; to calculate h.

Solution. Let $\gamma = \angle ACB$. $\quad \cos \gamma = AC \div BC.$

But $\quad AC = h \cot \alpha; \quad BC = h \cot \beta.$

$$\therefore \quad \cos \gamma = \frac{\cot \alpha}{\cot \beta},$$

from which γ may be found.

Knowing $\angle \gamma$ and m, we can calculate either AC or BC, and then h. Thus:

$$AC = m \cot \gamma; \quad h = AC \tan \alpha = m \cot \gamma \tan \alpha.$$

Our scheme for logarithmic calculation would be:

$$\begin{aligned} \log \cos \gamma &= \log \cot \alpha - \log \cot \beta; \quad \gamma = ? \\ \log h &= \log m + \log \cot \gamma + \log \tan \alpha; \quad h = ? \end{aligned}$$

Exercises.

1. Calculate h when $m = 1575$ feet; $\alpha = 32°$; $\beta = 19°$.
2. Calculate h when $m = 236.7$ feet; $\alpha = 58° 16'$; $\beta = 40° 34'$.
3. From a point A on level ground due south of an aeroplane, its angle of elevation is $41° 12'$; from a point B 1000 feet due east of A, the angle of elevation is $36° 41'$; how high is the aeroplane?

Part II. Vectors

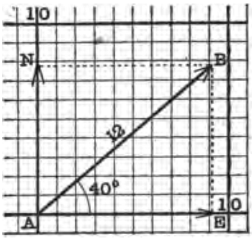
34. Vectors and their Components.

Definitions. A vector is a directed line-segment; that is, a line whose *length* and *direction* are given.

A **vector quantity** is any quantity capable of being represented by a directed line-segment, or vector.

Velocities and forces in Physics and Mechanics are vector quantities.

Velocities as Vectors. Suppose a ship to be sailing at the rate of 12 knots an hour in a direction E. 40° N. (figure). Let A mark its position at any moment; draw the directed line-segment AB so that its direction shall show the direction in which the ship is sailing, and its length (on an indicated scale) the speed of the ship. Then AB is a vector representing the velocity of the ship.



Note. Velocity is here used to include both *speed* or *rate* of motion, and *direction* of motion.

The motion of a ship may be resolved into two **components**, an easterly component AE, and a northerly component AN.

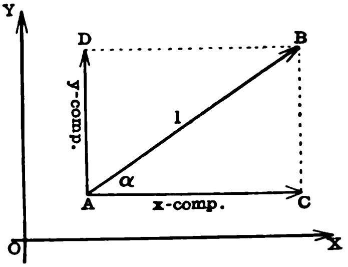
By inspection of the figure:

$$AE = 9.2 \text{ knots}; \quad AN = 7.8 \text{ knots}.$$

By use of the right triangles:

$$AE = 12 \cos 40° = 9.192 \text{ knots}; \quad AN = 12 \sin 40° = 7.714 \text{ knots}.$$

In this way any vector can be resolved into two components at right angles to each other, as indicated in the adjacent figure. Here AB is a given vector, and OX and OY are two rectangular reference lines.



Let l be the length of the vector and α its angle with OX. Then

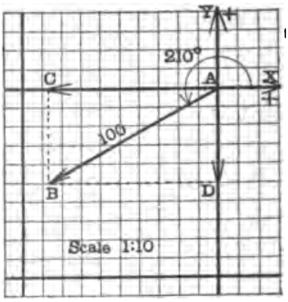
x-component of vector $= AC = l \cos \alpha$:
y-component of vector $= AD = l \sin \alpha$.

When the components AC and AD are given, we have

length of vector: $l = \sqrt{\textit{sum of squares of components}}$.

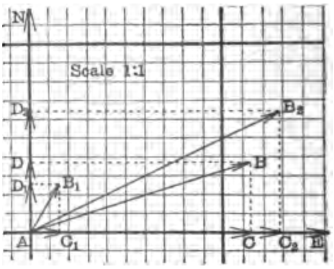
angle of vector: $\tan \alpha = \dfrac{\textit{y-component}}{\textit{x-component}}$.

When α is an obtuse angle, $l \cos \alpha$ will be negative; the meaning of the negative sign is that the x-component is directed toward the *left,* provided the positive direction has been chosen toward the *right.* Similarly for "up" and "down." Thus in the adjacent figure the components of AB are:



$$AC = 100 \cos 210° = -86.6 +,$$
$$AD = 100 \sin 210° = -50.0.$$

35. Sum of Vectors. Resultants. — Suppose now that a ship which is steaming E. 20° N. at 12 knots an hour is also being carried by an ocean current which flows in the direction N. 30° E. at 3 knots an hour. What is the actual motion of the ship?



Let AB and AB_1 be vectors representing the velocities of ship and current respectively. Let AC and AC_1 be their easterly components. Then

$$
\begin{aligned}
&\textit{total easterly speed of ship} &&= AC + AC_1 = AC_2.\\
\text{Likewise: }&\textit{total northerly speed of ship} &&= AD + AD_1 = AD_2.\\
\text{Also: }&\textit{resultant speed} &&= \sqrt{(AC_2)^2 + (AD_2)^2};\\
&\textit{resultant direction: } \tan \angle\, C_2AB_2 &&= \frac{C_2B_2}{AC_2} = \frac{AD_2}{AC_2}.
\end{aligned}
$$

The vector AB_2 is called the *sum* or *resultant* of the vectors AB and AB_1. Its x-component is the sum of the x-components of AB and AB_1. Similarly for the y-components. Likewise for more than two vectors.

Definition. *The* **sum** *or* **resultant** *of any number of vectors is a vector such that*

its x-component = sum of x-components of the given vectors;
its y-component = sum of y-components of the given vectors.

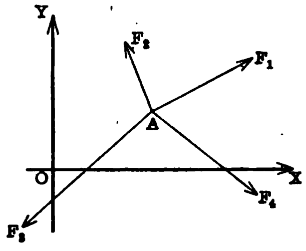
Exercises.

1. With the numerical values given above draw a figure to an accurate scale, not too small. Scale off and also calculate all the quantities used in the illustration.

2. Show that the resultant vector AB_2 is the diagonal of a parallelogram on AB and AB_1 as sides.

3. If a moving body is subjected to any number of motions, each of given amount and direction, explain how to find the resultant speed and direction.

36. Forces as Vectors. — Suppose a particle at A to be pulled upon by several forces, all in the same plane, as AF_1, AF_2, AF_3, AF_4 in the figure. Here each force is represented by a vector, showing the amount and direction of the pull.



What must be the amount and direction of a single force which is equivalent to the four given forces. This is called the **resultant** of the given system of forces.

Solution. Resolve each force into an x-component and a y-component. This is done by the formulas

$$x\text{-component} = |\mathit{force}| \times \cos\alpha; \quad y\text{-component} = |\mathit{force}| \times \sin\alpha;$$

here $|\mathit{force}|$ denotes the *magnitude* of the force, and α is the angle between OX and AF, measured in the counter-clockwise direction Thus for AF_4, $\alpha = 320°$ nearly.

Form the sum of the x-components, each with its proper sign, for a "total x-component," Similarly for the y-components. Then

$$\textit{Amount of Resultant Force} = \sqrt{(\textit{total x-comp.})^2 + (\textit{total y-comp.})^2};$$

$$\textit{Angle of Resultant Force:}\ \tan\alpha = \frac{\textit{total y-component}}{\textit{total x-component}}.$$

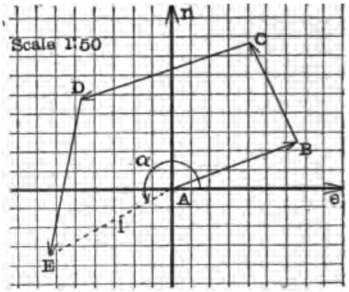
Exercises. Calculate the resultant of each of the following systems of forces. A force of n in the direction α is indicated by the symbol (n, α). Draw accurate figures.

1. (30 lb., 25°); (40 lb., 50°).
2. (25 lb., 40°); (18 lb., 70°); (35 lb., 160°).
3. (75 lb., 65°); (60 lb., 130°); (85 lb., 230°); (40 lb., 340°).
4. Show that the resultant of two forces is represented by the diagonal of a parallelogram whose sides represent the two forces.

37. Surveying. — As a last illustration of the use of vectors, suppose a surveyor to start from A and run the following lines:

A to B, E. 20° N., 345 ft.;
B to C, N. 25° W., 288 ft.;
C to D, W. 18° S., 467 ft.;
D to E, S. 2° W., 424 ft.

How far and in what direction is he now from his starting point?



In the figure each measurement is represented by a vector of proper length and direction. The side of one square represents 50 feet. We must determine l, the length of AE, and $\alpha = \angle\, eAE$.

To do this we calculate:

northerly component of AE
= *sum of northerly components of measured lines;*
easterly component of AE
= *sum of easterly components of measured lines.*

Here a negative northerly component is to be counted as a southerly component, and a negative easterly component is to be counted as a westerly component. Then

$$AE = \sqrt{(\textit{sum of north-comp's})^2 + (\textit{sum of east-comp's})^2};$$

$$\tan\alpha = \frac{\textit{sum of north-components}}{\textit{sum of east-components}}.$$

Exercise. Draw a figure to scale, showing the above data, and measure AE and α. Also calculate AE and α.

38. Exercises.

1. The hypotenuse of a right triangle is 500 ft. long, and one of its acute angles is 28° 32′. Show that the perpendicular from the vertex of the right angle to the hypotenuse is 209.9 ft.

2. If in Exercise 1 the hypotenuse is c ft. and the angle is α, show that the perpendicular is $c \sin \alpha \cos \alpha$.

3. From a point in the same horizontal plane with the foot of a tower the angle of elevation of its top is 11° 29′. From a point 100 ft. nearer to the foot of the tower the angle is 13° 18′. Find the height of the tower. *Ans.* 144.5 ft.

4. From one bank of a river the angle of elevation of the top of a tree on the opposite bank is 40° 22′. On moving back 120 ft., the angle of elevation is 29° 37′. Find the height of the tree and the width of the river.

5. At a certain point in the same horizontal plane with the foot of a column 25 ft. high, the angle of elevation of its top is 50°. What will be the angle of elevation at a point 15 ft. farther away? *Ans.* 34° 48′.

6. A column 75 ft. high stands on a pedestal 25 ft. high. From a certain point on the ground in the same horizontal plane with the foot of the pedestal, the latter subtends an angle of 15°. What angle does the column subtend at this point? *Ans.* 31° 58′.5.

7. A vertical pole 30 ft. long, and standing on level ground, casts a shadow 50 ft. long. What will be the length of the shadow when the sun is 10° higher?

8. The equatorial radius of the earth is 3960 miles. Find the radius and circumference of the 40th parallel of latitude. Find the radius of the arctic circle.

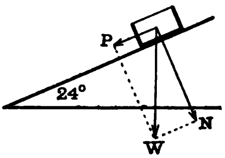
9. A billiard table whose dimensions are 36 in. by 72 in. stands with its long axis north and south, and a ball rests against the middle point of its southern cushion. Where should the ball be made to strike the western cushion so as to be deflected to the northeast corner?

10. If a ship steams N. 25° W. at 18 knots an hour in a current which moves S. 35° W. at 4 knots an hour, and a passenger walks across the deck at right angles to the length of the ship at the rate of 3 knots an hour, determine the resultant motion of the passenger, in speed and direction.

11. A ship leaves Boston light and sails E. 15° S., 25 miles; then E. 20° N., 40 miles; then N. 35° E., 60 miles; in what direction should she now sail to return directly to the starting point? How far will she have to go? (Consider the surface of the ocean as a plane.)

12. A weight of 100 lb. rests on a rough plane inclined at an angle of 24° to the horizontal. What is the pressure of the weight against the plane? (This pressure is due to the component of the weight perpendicular to the plane.)

If it takes a pull of 20 lb. parallel to the plane to overcome friction and start the weight moving, will it slide down the plane or remain at rest? (Calculate the component of the weight parallel to the plane and note whether it is more or less than 20 lb.)



Part III. Harmonic Analysis

39. Simple Waves. — The graph of the function

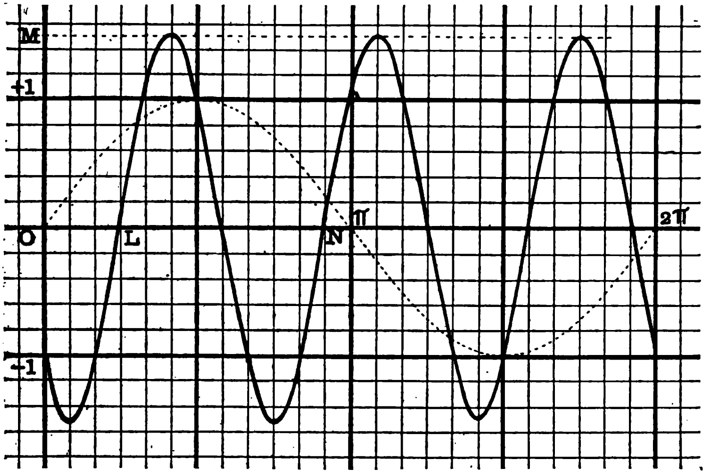
$$y = \sin x$$

is a wave curve of the simplest type, as shown in the figure on p. 13.

Such a curve may be altered in several ways without destroying its simple wave form. We may change

(a) the height of the crests, or **amplitude** of the wave;
(b) the **length** of the wave;
(c) the **phase** of the wave, depending on where it cuts the x-axis.

In this way we would get a wave like that in the following figure, where the original sine wave is shown for comparison.



Full line, $y = 1.5 \sin (3x - 4)$.
Amplitude $= OM = 1.5$.
Wave length $= LN = \frac{2\pi}{3}$ radians.
Phase $= OL = \frac{4}{3}$ radians.

Dotted line, $y = \sin x$.
Amplitude $= 1$.
Wave length $= 2\pi$ radians.
Phase $= 0$.

The most general expression for the simple wave which results when all the above changes have been made in the wave for sin x, is

$$y = k \sin (ax + b).$$

Example. $y = 1.5 \sin (3x - 4)$. (See preceding figure.)

(*a*) *The amplitude.* The greatest value of $\sin (3x - 4)$ is 1, since the sine function cannot exceed this value; hence the greatest value of y is 1.5. This shows the height of the wave; that is,

$$\textit{amplitude} = 1.5.$$

(*b*) *The wave-length.* This is determined by finding the points where the wave crosses the x-axis. These are marked by the values of x for which $\sin (3x - 4) = 0$. But this is zero when the angle $(3x - 4)$ is an integral multiple of π;

$$\sin (3x - 4) = 0 \quad \text{if} \quad 3x - 4 = m\pi, \quad \text{or} \quad x = \frac{m\pi + 4}{3} \text{ radians},$$

where m is any whole number.

Putting $m = 0, 1, 2, 3, \ldots$, we get the successive crossing points:

$$x_0 = \frac{4}{3}; \quad x_1 = \frac{\pi + 4}{3}; \quad x_2 = \frac{2\pi + 4}{3}; \quad x_3 = \frac{3\pi + 4}{3}; \quad \text{etc.}$$

These values in degrees are, very nearly,

$$x_0 = 76°.6; \quad x_1 = 136°.6; \quad x_2 = 196°.6; \quad x_3 = 256°.6; \quad \text{etc.}$$

The distance between alternate crossing points, as x_0 to x_2, is the wave-length:

$$\textit{wave-length} = x_2 - x_0 = \frac{2\pi + 4}{3} - \frac{4}{3} = \frac{2\pi}{3} \text{ radians.}$$

This is one-third of the wave-length of the fundamental sine wave.

(*c*) *The phase.* This is determined by the first crossing point, x_0. Hence

$$\textit{phase} = x_0 = OL = \frac{4}{3} \textit{ radians.}$$

In general, for the wave $y = k \sin (ax + b)$,

$$\textit{amplitude} = k;$$

$$\textit{wave-length} = \frac{2\pi}{a} \textit{ radians};$$

$$\textit{phase} = -\frac{b}{a} \textit{ radians.}$$

Exercises. Draw the following waves, showing each in comparison with $y = \sin x$.

1. $y = 2 \sin (x - 1)$.

2. $y = 3 \sin (2x - 3)$.

3. $y = 2.5 \sin (2x + 3)$.

4. $y = 4 \sin (3x - 60°)$.

5. Prove the statements made above regarding $y = k \sin (ax + b)$.

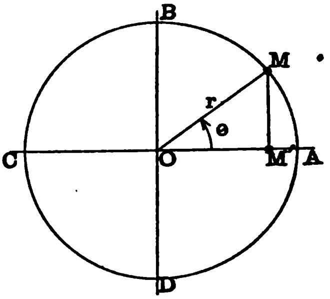
6. Draw the graph of $e = 110 \sin (240\pi t - \pi)$.

This equation describes the rise and fall of the electromotive force at a fixed point in an ideal alternating current circuit.

Here e and t take the place of y and x; e stands for electromotive force in volts, t for the time in seconds. Show that the greatest value of the electromotive force is 110 volts, and that there will be 120 vibrations per second.

In drawing the graph, use care in the choice of scales. Thus on cross-section paper, one square of the vertical scale might be taken to represent 10 volts, and 10 squares of the horizontal scale might be taken to represent $\frac{1}{120}$ seconds.

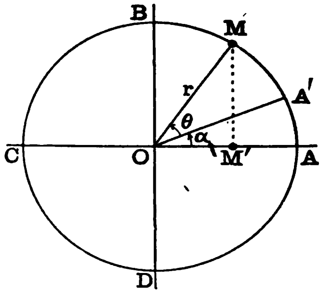
40. Simple Harmonic Motion. — Referring again to the illustration of (**24**), the figure for which is reproduced here, we see that, as M moves around the circle, its projection M' moves back and forth along AC.



When $\theta = 0$, M and M' are together at A. Then, if we suppose θ to increase uniformly, M will move around the circle with uniform speed; but M' will move along AC with variable speed, slowly at first, then faster until it reaches O, when its speed will be greatest, then more slowly until it reaches C, where it will come to rest and start back toward A. This type of motion is called *simple harmonic motion.*

A body which has this motion vibrates back and forth past a middle position with variable speed. The distance of the body from its mid-position is called its **displacement.** From the figure,

$$\textit{displacement of } M' = d = OM' = r \cos \theta.$$



When $\theta = 0$, d reaches its greatest value r, which is called the *amplitude* of the vibration.

If we measure θ from some other fixed radius OA' in place of OA, we shall have

$$d = r \cos (\theta + \alpha).$$

The greatest value of d is now reached when $\theta + \alpha = 0$, or when $\theta = -\alpha$; this angle is called the *phase* of the vibration.

If we suppose the angular speed of the radius OM to be ω radians per second, and t to represent the time in seconds elapsed since M was at A, then $\theta = \omega t$, and

$$d = r \cos (\omega t + \alpha).$$

This is an equation of the type

$$y = k \cos (ax + b),$$

and, like the equation

$$y = k \sin (ax + b),$$

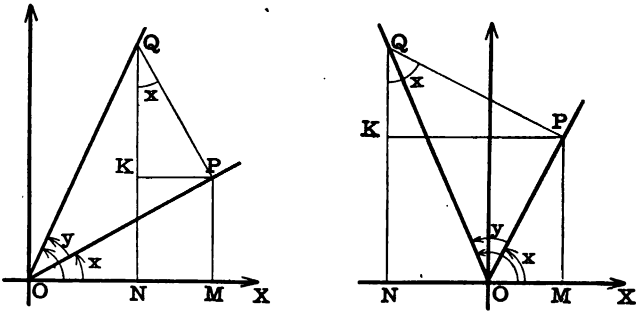
is represented graphically by a simple wave curve.

Exercise. Draw the graph of $d = r \cos (\omega t + \alpha)$, when $r = 5$, $\omega = 3$, and $\alpha = \frac{\pi}{3}$.

CHAPTER III

FUNCTIONS OF SEVERAL ANGLES

41. Formulas for sin $(x+y)$ and cos $(x+y)$. — Let x and y be two angles, each of which we first assume to be less than 90°. Their sum will then fall in the first or the second quadrant. The two cases are illustrated in the figures, and the demonstration which follows applies to either figure.



Construct $\angle\, XOP = x$ and $\angle\, POQ = y$, the terminal side of x being taken as the initial side of y.

From Q, any point on the terminal side of y, draw perpendiculars NQ and PQ to the sides of angle x, produced if necessary. Draw $MP \perp OX$ and $KP \perp NQ$.

Then $\angle\, KQP = x$, and in either figure,

$$\sin(x+y) = \frac{NQ}{OQ} = \frac{MP+KQ}{OQ} = \frac{MP}{OQ} + \frac{KQ}{OQ}$$

$$= \frac{MP}{OP}\cdot\frac{OP}{OQ} + \frac{KQ}{PQ}\cdot\frac{PQ}{OQ}.$$

Hence

(a) $$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

Also, noting that ON in the second figure is a negative line,

$$\cos(x+y)=\frac{ON}{OQ}=\frac{OM-NM}{OQ}=\frac{OM}{OQ}-\frac{KP}{OQ}$$
$$=\frac{OM}{OP}\cdot\frac{OP}{OQ}-\frac{KP}{PQ}\cdot\frac{PQ}{OQ}.$$

Hence

(b) $$\cos(x+y)=\cos x\cos y-\sin x\sin y.$$

42. Generalization of Formulas (*a*) and (*b*). — In the preceding proofs we have assumed angles x and y less than 90°. By trial it will be found that the same proofs hold for figures drawn like those above with any two angles x and y. We shall not do this, but instead, shall use the method of induction to prove that the formulas are always true. That is, we shall show that if the formulas are true for any two given angles, they will remain true when either angle is increased by 90°.

Let A and B be any two angles and assume that

(a') $\sin(A+B)=\sin A\cos B+\cos A\sin B,$

(b') $\cos(A+B)=\cos A\cos B-\sin A\sin B.$

In (a') change A to $A+90°=A'$; then prove that

(a'') $\sin(A'+B)=\sin A'\cos B+\cos A'\sin B.$

The left member of (a'') reduces to

$$\sin(A'+B)=\sin(A+B+90°)=\cos(A+B). \quad (17)$$

The right member of (a''), with A' replaced by $A+90°$, gives

$$\sin(A+90°)\cos B+\cos(A+90°)\sin B$$
$$=\cos A\cos B-\sin A\sin B. \quad (17)$$

Note. By drawing a figure with an angle A in any quadrant, and then drawing the angle $A+90°$, it can be seen directly that

$$\sin(A+90°)=\cos A, \quad \text{and} \quad \cos(A+90°)=-\sin A.$$

Hence (a'') follows from (b'), which was assumed to be true.

If in (a') we replace B by $B+90°$, it follows in precisely the same way that the new equation is true.

Similarly it can be shown that (b') remains true when either A or B is increased by 90°.

We have now shown that *if formulas* (a') *and* (b') *are true in any given case, they are still true when either angle is increased by* 90°.

But they are true when $A = x$ and $B = y$, x and y being acute angles; hence they are true when $A = x + 90°$; if true for $A = x + 90°$, they are true for $A = x + 2 \cdot 90°$; and so on. Similarly for the angle B.

In a similar manner it can be shown that formulas (a) and (b) remain true when either angle is diminished by 90°.

Using the letters x and y in place of A and B, formulas (a) and (b) are true for all values of x and y, positive or negative.

43. Formulas for sin $(x - y)$ and cos $(x - y)$. — Replacing y by $-y$ in (a) and (b), we have the two equations

$$\sin(x - y) = \sin x \cos(-y) + \cos x \sin(-y);$$
$$\cos(x - y) = \cos x \cos(-y) - \sin x \sin(-y).$$

But $\cos(-y) = \cos y$ and $\sin(-y) = -\sin y$. **(18)**

Therefore the two preceding equations become

$$\text{(c)} \qquad \boldsymbol{\sin(x - y) = \sin x \cos y - \cos x \sin y;}$$
$$\text{(d)} \qquad \boldsymbol{\cos(x - y) = \cos x \cos y + \sin x \sin y.}$$

Equations (a), (b), (c), (d) are usually called the addition and subtraction formulas of trigonometry. All the other working formulas are deduced from them.

44. Formulas for tan $(x \pm y)$ and cot $(x \pm y)$. — Dividing (a) by (b), we have

$$\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$
$$= \frac{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}}{1 - \dfrac{\sin x \sin y}{\cos x \cos y}}.$$

Hence,

$$\text{(e)} \qquad \boldsymbol{\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}}.$$

Similarly,

$$\text{(f)} \qquad \boldsymbol{\cot(x + y) = \frac{\cot x \cot y - 1}{\cot x + \cot y}}.$$

Also, from (c) and (d), by division,

$$\text{(g)} \qquad \boldsymbol{\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}}.$$

$$\text{(h)} \qquad \boldsymbol{\cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}}.$$

Exercises.

1. If $\sin x = \frac{2}{5}$ and $\sin y = \frac{4}{5}$, calculate $\sin(x + y)$.

Four answers: $\frac{2}{25}[\pm 3 \pm 2\sqrt{21}.]$

2. If $\cos x = \frac{3}{5}$ and $\cos y = \frac{9}{41}$, calculate $\cos(x + y)$.
3. If $\sin\alpha = \frac{12}{13}$ and $\sin\beta = \frac{5}{13}$, calculate $\cos(\alpha - \beta)$.

Show that,

4. $\sin(60° + x) - \sin(60° - x) = \sin x.$
5. $\cos(45° + \theta) - \cos(45° - \theta) = -\sqrt{2}\sin\theta.$
6. $\cot\theta + \tan\phi = \dfrac{\cos(\theta - \phi)}{\sin\theta\cos\phi}.$
7. $\cos(A - 45°) - \sin(A + 45°) = 0.$
8. $\sin n\theta\cos\theta + \cos n\theta\sin\theta = \sin(n + 1)\theta.$
9. $\tan\left(\theta + \frac{\pi}{4}\right) + \cot\left(\theta - \frac{\pi}{4}\right) = 0.$
10. $\dfrac{\sin(\alpha - \beta)}{\sin\alpha\sin\beta} = \cot\beta - \cot\alpha.$
11. $\dfrac{\tan(45° + \theta)}{\tan(45° - \theta)} = \dfrac{(1 + \tan\theta)^2}{(1 - \tan\theta)^2}.$
12. From the functions of 30° and 45° calculate the functions of 75°.
13. From the functions of 45° and 60° calculate the functions of 15° and those of 105°.
14. From the functions of 135° and 300° calculate the functions of 165°.

For convenience we collect formulas (a), (b) . . . , (h) and form Group B, numbering them consecutively with the formulas of Group A. The formulas for cot $(x \pm y)$ may be omitted; in place of them use the formulas for tan $(x \pm y)$ with the fractions inverted.

Formulas, Group B.

$$\sin(x + y) = \sin x\cos y + \cos x\sin y. \tag{9}$$

$$\cos(x + y) = \cos x\cos y - \sin x\sin y. \tag{10}$$

$$\sin(x - y) = \sin x\cos y - \cos x\sin y. \tag{11}$$

$$\cos(x - y) = \cos x\cos y + \sin x\sin y. \tag{12}$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}. \tag{13}$$

$$\cot(x + y) = \frac{\cot x\cot y - 1}{\cot x + \cot y}. \tag{14}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}. \tag{15}$$

$$\cot(x - y) = \frac{\cot x\cot y + 1}{\cot y - \cot x}. \tag{16}$$

45. Functions of $2x$. — Putting $y = x$ in (9), (10) and (13) of Group B, we have

$$\sin 2x = 2 \sin x \cos x, \tag{17}$$

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x, \\ &= 1 - 2 \sin^2 x, \\ &= 2 \cos^2 x - 1. \end{aligned} \tag{18}$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}. \tag{19}$$

For $\cot 2x$ use $\dfrac{1}{\tan 2x}$.

Exercises.

1. Verify these formulas when x is 30°; 45°; 150°; − 60°.

Show that,

2. $\sin 4x = 2 \sin 2x \cos 2x$.
3. $\cos 4x = 1 - 2 \sin^2 2x = 1 - 8 \sin^2 x \cos^2 x$.
4. $\dfrac{1 - \cos 2x}{\sin 2x} = \tan x$.
5. $\cot x - \tan x = 2 \cot 2x$.
6. Calculate the functions of $2x$ when $\sin x = \frac{5}{13}$.
   Ans. $\sin 2x = \pm \frac{120}{169}$; etc.
7. Calculate the functions of $2x$ when $\tan x = \frac{4}{3}$.

46. Functions of $\frac{1}{2}x$. — The second and third values of $\cos 2x$ in (18) are

$$\cos 2x = 1 - 2 \sin^2 x,$$
$$\cos 2x = 2 \cos^2 x - 1.$$

Solving these for $\sin x$ and $\cos x$ respectively, we have

$$\sin x = \pm \sqrt{\frac{1 - \cos 2x}{2}}, \quad \cos x = \pm \sqrt{\frac{1 + \cos 2x}{2}}.$$

Replacing x by $\frac{1}{2}x$, these become

$$\sin \tfrac{1}{2}x = \pm \sqrt{\frac{1 - \cos x}{2}}, \tag{20}$$

$$\cos \tfrac{1}{2}x = \pm \sqrt{\frac{1 + \cos x}{2}}. \tag{21}$$

Dividing (20) by (21),

$$\tan \tfrac{1}{2}x = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}. \tag{22}$$

The second of these forms is obtained by multiplying both sides of the fraction under the radical sign by $1 - \cos x$. This gives

$$\tan \tfrac{1}{2} x = \pm\sqrt{\frac{(1-\cos x)^2}{1-\cos^2 x}} = \pm\sqrt{\frac{(1-\cos x)^2}{\sin^2 x}} = \frac{1-\cos x}{\sin x}.$$

This fraction always has the same sign as $\tan \frac{1}{2} x$, so the sign $\pm$ has been dropped. The third form for $\tan \frac{1}{2} x$ comes from using the multiplier $1 + \cos x$ instead of $1 - \cos x$.

Formulas, Group C.

$$(17)\quad \sin 2x = 2 \sin x \cos x.$$

$$(18)\quad \cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1.$$

$$(19)\quad \tan 2x = \frac{2\tan x}{1-\tan^2 x}.$$

$$(20)\quad \sin \tfrac{1}{2}x = \pm\sqrt{\frac{1-\cos x}{2}}.$$

$$(21)\quad \cos \tfrac{1}{2}x = \pm\sqrt{\frac{1+\cos x}{2}}.$$

$$(22)\quad \tan \tfrac{1}{2}x = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}.$$

Exercises.

1. Calculate the functions of 15° from those of 30°.
2. Calculate the functions of $22\frac{1}{2}$° from those of 45°.
3. Calculate $\tan \frac{1}{2} x$ when $\cos x = \frac{3}{5}$. Use all three forms.
4. Calculate the values of $\tan (2x - y)$, when $\sin x = \frac{4}{5}$ and $\cos y = \frac{5}{13}$.

Show that,

5. $\sin 6x = 2 \sin 3x \cos 3x$.
6. $\sec 2x = \dfrac{\sec^2 x}{2 - \sec^2 x}$.
7. $\dfrac{1-\sin 2x}{\cos 2x} = \dfrac{1-\tan x}{1+\tan x}$.
8. $\cos^4 \theta - \sin^4 \theta = \cos 2\theta$.
9. $\dfrac{\cos^3\theta + \sin^3\theta}{\cos\theta + \sin\theta} = \dfrac{2-\sin 2\theta}{2}$.
10. $\csc x - \tan x = \tan \frac{1}{2} x$.
11. $(\sin \frac{1}{2}\theta - \cos \frac{1}{2}\theta)^2 = 1 - \sin\theta$.
12. $\dfrac{1+\sec\theta}{\sec\theta} = 2 - 2\sin^2 \frac{1}{2}\theta$.
13. $\sin\beta \cot \frac{1}{2}\beta = 2\cos^2 \frac{1}{2}\beta$.
14. $2 + \tan\beta \tan \frac{1}{2}\beta = \dfrac{2}{1-\tan^2 \frac{1}{2}\beta}$.
15. $2\cot\beta \cot\dfrac{\beta}{2} = \cot^2\dfrac{\beta}{2} - 1$.
16. $\dfrac{\cos\beta}{1-\sin\beta} = \dfrac{1+\tan\dfrac{\beta}{2}}{1-\tan\dfrac{\beta}{2}}$.

47. Formulas for sin $u \pm$ sin v and for cos $u \pm$ cos v. — Formulas (9) to (12) of Group B are

$$\begin{aligned}\sin(x+y) &= \sin x\cos y + \cos x\sin y,\\ \sin(x-y) &= \sin x\cos y - \cos x\sin y,\\ \cos(x+y) &= \cos x\cos y - \sin x\sin y,\\ \cos(x-y) &= \cos x\cos y + \sin x\sin y.\end{aligned}$$

Forming the sum and difference, respectively, of the first two equations, we have

$$\text{(p)}\qquad \sin(x+y) + \sin(x-y) = 2\sin x\cos y;$$

$$\text{(q)}\qquad \sin(x+y) - \sin(x-y) = 2\cos x\sin y.$$

Forming the sum and difference, respectively, of the other two equations, we have

$$\text{(r)}\qquad \cos(x+y) + \cos(x-y) = 2\cos x\cos y;$$

$$\text{(s)}\qquad \cos(x+y) - \cos(x-y) = -2\sin x\sin y.$$

Now in the last four equations let

$$x + y = u \quad\text{and}\quad x - y = v.$$

Then

$$x = \frac{u+v}{2} \quad\text{and}\quad y = \frac{u-v}{2}.$$

Substituting in equations (p), (q), (r), (s), we have four formulas, called the *addition theorems* of trigonometry, namely

Formulas, Group D.

$$\text{(23)}\qquad \sin u + \sin v = 2\sin\frac{u+v}{2}\cos\frac{u-v}{2}.$$

$$\text{(24)}\qquad \sin u - \sin v = 2\cos\frac{u+v}{2}\sin\frac{u-v}{2}.$$

$$\text{(25)}\qquad \cos u + \cos v = 2\cos\frac{u+v}{2}\cos\frac{u-v}{2}.$$

$$\text{(26)}\qquad \cos u - \cos v = -2\sin\frac{u+v}{2}\sin\frac{u-v}{2}.$$

The four equations (*p*), (*q*), (*r*), (*s*) are themselves often considered as a group of formulas, and are repeated below, with right and left members interchanged.

Formulas, Group D′

(23′) $$2 \sin x \cos y = \sin (x + y) + \sin (x - y).$$

(24′) $$2 \cos x \sin y = \sin (x + y) - \sin (x - y).$$

(25′) $$2 \cos x \cos y = \cos (x + y) + \cos (x - y).$$

(26′) $$-2 \sin x \sin y = \cos (x + y) - \cos (x - y).$$

Example 1. Show that $\dfrac{\sin x + \sin y}{\sin x - \sin y} = \dfrac{\tan \frac{x+y}{2}}{\tan \frac{x-y}{2}}$.

$$\frac{\sin x + \sin y}{\sin x - \sin y} = \frac{2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}}{2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}}$$

$$= \tan \frac{x+y}{2} \cot \frac{x-y}{2} = \frac{\tan \frac{x+y}{2}}{\tan \frac{x-y}{2}}.$$

Example 2. Show that $\dfrac{\cos 75° + \cos 15°}{\cos 75° - \cos 15°} = -\sqrt{3}$.

$$\frac{\cos 75° + \cos 15°}{\cos 75° - \cos 15°} = \frac{2 \cos 45° \cos 30°}{-2 \sin 45° \sin 30°} = -\cot 45° \cot 30° = -\sqrt{3}.$$

Exercises. Show that:

1. $\sin 3x + \sin 5x = 2 \sin 4x \cos x$.
2. $\sin 10\theta + \sin 6\theta = 2 \sin 8\theta \cos 2\theta$.
3. $\cos 2x + \cos 4x = 2 \cos 3x \cos x$.
4. $\sin 7\alpha - \sin 5\alpha = 2 \cos 6\alpha \sin \alpha$.
5. $\cos 4\theta - \cos 6\theta = 2 \sin 5\theta \sin \theta$.
6. $\cos x + \cos 2x = 2 \cos \frac{3x}{2} \cos \frac{x}{2}$.
7. $\sin 30° + \sin 60° = \sqrt{2} \cos 15°$.
8. $\sin 70° - \sin 10° = \cos 40°$.
9. $\sin 5x \cos 3x = \frac{1}{2}(\sin 8x + \sin 2x)$.
10. $2 \cos 10° \sin 50° = \sin 60° + \sin 40°$.
11. $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan \dfrac{A+B}{2}$.
12. $\dfrac{\sin \theta + \sin 3\theta}{\cos \theta + \cos 3\theta} = \tan 2\theta$.
13. $2 \cos \alpha \cos \beta = \cos (\alpha - \beta) + \cos (\alpha + \beta)$.
14. $\sin 4\theta \sin \theta = \frac{1}{2}(\cos 3\theta - \cos 5\theta)$.
15. $\cos 8x - \cos 4x = -4 \sin 2x \sin 3x \cos 3x$.
16. $\sin (2x + 3y) + \sin (2x - 3y) = 2 \sin 2x \cos 3y$.

All the formulas derived above should be learned in *verbal form* rather than in terms of any particular letters. Thus:

Formula (9).

The sine of the sum of two angles equals sine of first times cosine of second plus cosine of first times sine of second.

Formula (13).

The tangent of the sum of two angles equals the sum of their tangents divided by 1 *minus the product of their tangents.*

Formula (17).

The sine of any angle equals twice the sine of half the angle times the cosine of half the angle.

Formula (20).

The sine of any angle equals plus or minus the square root of 1 *minus the cosine of twice the angle divided by* 2.

Formula (23).

The sum of two sines equals twice the sine of half the sum of the angles times the cosine of half their difference.

48. Exercises involving the use of formulas (1) to (26).

1. If $\sin x = \frac{4}{5}$ and $\sin y = \frac{3}{5}$, find the value of $\sin(x + y)$ and $\cos(x + y)$ when x and y are both in the first quadrant.

2. As in exercise 1, when x and y are both in the second quadrant.

3. If $\cos x = \frac{3}{5}$ and $\cos y = \frac{9}{41}$, calculate $\sin(x + y)$ and $\cos(x + y)$ when x and y are both in the first quadrant.

4. As in exercise 3, when x and y are both in the fourth quadrant.

5. If $\sin x = \frac{1}{3}$ and $\sin y = \frac{2}{3}$, calculate all values of $\sin(x \pm y)$.

6. If $\sin \alpha = \frac{3}{4}$ and $\sin \beta = \frac{3}{5}$, calculate all values of $\cos(\alpha \pm \beta)$.

7. If $\cos \alpha = \frac{3}{4}$ and $\cos \beta = \frac{2}{5}$, calculate all values of $\tan(\alpha \pm \beta)$.

8. Calculate $\sin(x + y + z)$ when $\sin x = \frac{12}{13}$, $\sin y = \frac{24}{25}$, $\sin z = \frac{40}{41}$, and x, y, z all lie in the first quadrant.

9. As in exercise 8, when x, y, z all lie in the second quadrant.

10. Calculate $\cos(x + y + z)$ when $\cos x = \frac{3}{5}$, $\cos y = \frac{5}{13}$, $\cos z = \frac{7}{25}$, and x, y, z all lie in the first quadrant.

11. As in exercise 10, when x, y, z all lie in the fourth quadrant.

12. Calculate $\tan(x + y)$ when $\tan x = \sqrt{3}$ and $\cot y = \sqrt{3}$.

13. Calculate all values of $\sin 2(x - y)$ and of $\tan(2x - y)$ when $\tan x = \frac{4}{3}$ and $\tan y = \frac{12}{5}$.

14. Calculate all values of $\cos(\alpha + \beta)$ when $\tan \alpha = m$ and $\tan \beta = n$.

15. Calculate $\cot(\alpha - \beta)$ when $\tan \alpha = a + 1$ and $\tan \beta = a - 1$.

16. Calculate $\tan(\alpha + \beta)$ when $\tan \alpha = \dfrac{x}{x + 1}$ and $\tan \beta = \dfrac{1}{2x + 1}$.

17. If $\tan \alpha = \frac{1}{7}$ and $\tan \beta = \frac{2}{11}$, calculate $\tan(2\alpha + \beta)$.

18. Calculate sin 75°, cos 75°, and tan 75°, by use of the relation (a) $75° = \dfrac{150°}{2}$; (b) $75° = 135° - 60°$.

19. Calculate the functions of $202\frac{1}{2}°$.

Prove the following identities:

20. $\sin x \sin (y - z) + \sin y \sin (z - x) + \sin z \sin (x - y) = 0.$

21. $\cos x \sin (y - z) + \cos y \sin (z - x) + \cos z \sin (x - y) = 0.$

22. $\cos (x + y) \cos (x - y) + \sin (y + z) \sin (y - z) - \cos (x + z) \cos (x - z) = 0.$

23. $\cos (x - y + z) = \cos x \cos y \cos z + \cos x \sin y \sin z$
$\qquad - \sin x \cos y \sin z + \sin x \sin y \cos z.$

24. $\sin 3x = 3 \sin x - 4 \sin^3 x.$

25. $\cos 3x = 4 \cos^3 x - 3 \cos x.$

26. $\tan 3x = \dfrac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}.$

27. $\cot 3x = \dfrac{\cot^3 x - 3 \cot x}{3 \cot^2 x - 1}.$

28. $\tan 4\theta = \dfrac{4 \tan \theta \, (1 - \tan^2 \theta)}{1 - 6 \tan^2 \theta + \tan^4 \theta}.$

29. $\dfrac{\sin (\alpha + \beta)}{\cos \alpha \cos \beta} = \tan \alpha + \tan \beta.$

30. $\dfrac{\sin (\alpha + \beta)}{\sin \alpha \cos \beta} = \tan \alpha \tan \beta + 1.$

31. $\dfrac{\cos (\alpha - \beta)}{\sin \alpha \sin \beta} = \cot \alpha \cot \beta + 1.$

32. $\dfrac{\sin (\alpha - \beta)}{\cos \alpha \cos \beta} = \tan \alpha - \tan \beta.$

33. $\dfrac{\sin (x + y)}{\sin (x - y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}.$

34. $\dfrac{\cos (x + y)}{\cos (x - y)} = \dfrac{\cot y - \tan x}{\cot y + \tan x}.$

35. $\sin (\theta + \phi) \sin (\theta - \phi) = \cos^2 \phi - \cos^2 \theta.$

36. $\cos (u + v) \cos (u - v) = \cos^2 u - \sin^2 v.$

37. $\sin (A + 45°) = \dfrac{1}{\sqrt{2}} (\sin A + \cos A).$

38. $\cot \left(A - \dfrac{\pi}{4}\right) = \dfrac{\cot A + 1}{1 - \cot A}.$

39. $\tan \left(\theta - \dfrac{\pi}{4}\right) = \dfrac{\tan \theta - 1}{\tan \theta + 1}.$

40. $\tan \left(\dfrac{\pi}{4} + \theta\right) = \dfrac{1 + \tan \theta}{1 - \tan \theta}.$

41. $\tan \left(\alpha + \dfrac{\pi}{3}\right) + \tan \left(\alpha - \dfrac{\pi}{3}\right) = \dfrac{8 \cot \alpha}{\cot^2 \alpha - 3}.$

42. $\dfrac{\sin \dfrac{5\pi}{12}}{\sin \dfrac{\pi}{12}} + \dfrac{\cos \dfrac{5\pi}{12}}{\cos \dfrac{\pi}{12}} = 4.$

43. $\tan \left(\dfrac{\pi}{4} + \theta\right) = \dfrac{1}{\tan \left(\dfrac{\pi}{4} - \theta\right)}.$

44. $\cos \left(\theta + \dfrac{\pi}{4}\right) + \sin \left(\theta - \dfrac{\pi}{4}\right) = 0.$

45. $\cot \left(\theta + \dfrac{\pi}{4}\right) + \tan \left(\theta - \dfrac{\pi}{4}\right) = 0.$

46. $\cot \left(\theta - \dfrac{\pi}{4}\right) + \tan \left(\theta + \dfrac{\pi}{4}\right) = 0.$

47. $\cot \dfrac{\pi}{8} + \tan \dfrac{\pi}{8} = 2 \sqrt{2}.$

48. $2 \cos \dfrac{\pi}{8} = \sqrt{2 + \sqrt{2}}.$

49. $\sqrt{2} \sin (\theta - 45°) = \sin \theta - \cos \theta.$

50. $\sin 2x = \dfrac{2 \tan x}{1 + \tan^2 x}.$

51. $\sec 2x = \dfrac{\csc^2 x}{\csc^2 x - 2}.$

52. $\cot \theta - \cot 2\theta = \csc 2\theta.$

53. $\sec^2 \theta \cos 2\theta = 1 - \tan^2 \theta.$

54. $1 + \tan \theta \tan 2\theta = \sec 2\theta.$

55. $1 - \cos 2x = \tan x \sin 2x.$

56. $\sec 2\theta = \dfrac{1 + \tan^2 \theta}{1 - \tan^2 \theta}.$

57. $\dfrac{\sin 2\theta}{1 + \cos 2\theta} = \tan \theta.$

58. $\dfrac{\sin 2\theta}{1 - \cos 2\theta} = \cot \theta.$

59. $\cot^2\theta - 1 = 2\cot\theta\cot 2\theta.$

60. $2 - \sec^2\theta = \sec^2\theta\cos 2\theta.$

61. $\dfrac{\cos 2\theta}{1-\sin 2\theta} = \dfrac{1+\tan\theta}{1-\tan\theta}.$

62. $\dfrac{\cos 3x}{\cos x} = 2\cos 2x - 1.$

63. $\tan^2 x = \dfrac{1-\cos 2x}{1+\cos 2x}.$

64. $\dfrac{\cos 3\theta}{\sin\theta} + \dfrac{\sin 3\theta}{\cos\theta} = 2\cot 2\theta.$

65. $\dfrac{\tan\theta+\cot\theta}{\cot\theta-\tan\theta} = \sec 2\theta.$

66. $\tan(45° + \phi) - \tan(45° - \phi) = 2\tan 2\phi.$

67. $\dfrac{\cos^3\phi - \sin^3\phi}{\cos\phi - \sin\phi} = \dfrac{2+\sin 2\phi}{2}.$

68. $\dfrac{\cos^5\phi - \sin^5\phi}{\cos\phi - \sin\phi} = 1 + \frac{1}{2}\sin 2\phi - \frac{1}{4}\sin^2 2\phi.$

69. $\dfrac{\sin x + \cos x}{\cos x - \sin x} = \tan 2x + \sec 2x.$

70. $\sin 2x\tan 2x = \dfrac{4\tan^2 x}{1-\tan^4 x}.$

71. $\cos^2\theta + \sin^2\theta\cos 2\phi = \cos^2\phi + \sin^2\phi\cos 2\theta.$

72. $1 + \cos 2(\theta - \phi)\cos 2\phi = \cos^2\theta + \cos^2(\theta - 2\phi).$

73. $\dfrac{\tan^2\left(\theta+\frac{\pi}{4}\right)-1}{\tan^2\left(\theta+\frac{\pi}{4}\right)+1} = \sin 2\theta.$

74. $\dfrac{\cos\left(x+\frac{\pi}{4}\right)}{\cos\left(x-\frac{\pi}{4}\right)} = \sec 2x - \tan 2x.$

75. $\tan x = \dfrac{\sin x + \sin 2x}{1+\cos x+\cos 2x}.$

76. $\tan x = \dfrac{\sin 2x - \sin x}{1-\cos x+\cos 2x}.$

77. $\sec 2\theta - \frac{1}{2}\tan 2\theta\sin 2\theta = \dfrac{\cot^2\theta+\tan^2\theta}{\cot^2\theta-\tan^2\theta}$

78. $\dfrac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta} = \sqrt{\dfrac{1-\sin 2\theta}{1+\sin 2\theta}}.$

79. $\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2 = 1+\sin\theta.$

80. $\left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)^2 = 1-\sin\theta.$

81. $\dfrac{\cos\theta}{1+\sin\theta} = \dfrac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}.$

82. $\dfrac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} = \sec x + \tan x.$

83. $\tan x - \tan\frac{x}{2} = \tan\frac{x}{2}\sec x.$

84. $\dfrac{1+\sec\phi}{\sec\phi} = 2\cos^2\frac{\phi}{2}.$

85. $\sec^2\frac{x}{2} = 2\tan\frac{x}{2}\csc x.$

86. $\dfrac{1+\cos 3\phi}{\sin 3\phi} = \cot\dfrac{3\phi}{2}.$

87. $\dfrac{1+\sin 45°}{\cos 45°} = \tan 67\frac{1}{2}°.$

88. $\dfrac{1}{\sec\theta+\tan\theta} = \cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

89. $\dfrac{1+\sin x+\cos x}{1+\sin x-\cos x} = \cot\dfrac{x}{2}.$

90. $\tan\frac{x}{2} = \sqrt{\dfrac{2\sin x - \sin 2x}{2\sin x + \sin 2x}}.$

91. $\sqrt{3}\sin 75° - \cos 75° = \sqrt{2}.$

92. $\sin\dfrac{5\theta}{2}\cos\dfrac{\theta}{2} - \sin\dfrac{9\theta}{2}\cos\dfrac{3\theta}{2} + \cos 4\theta\sin 2\theta = 0.$

93. $\sin 4x + \sin 2x = 2\sin 3x\cos x.$

94. $\sin 3x + \sin 5x = 8 \sin x \cos^2 x \cos 2x.$

95. $\dfrac{\cot 15^\circ - \tan 15^\circ}{\cot 15^\circ + \tan 15^\circ} = \dfrac{1}{2}\sqrt{3}.$ **97.** $\cos 100^\circ - \cos 40^\circ = -\sin 20^\circ.$

96. $\dfrac{1 - \sqrt{2} \sin 75^\circ}{1 - \sqrt{2} \cos 75^\circ} = -\cot 60^\circ.$ **98.** $\sin\left(\dfrac{\pi}{3} + \alpha\right) - \sin\left(\dfrac{\pi}{3} - \alpha\right) = \sin \alpha.$

99. $\cos\left(\dfrac{\pi}{4} + \alpha\right) - \cos\left(\dfrac{\pi}{4} - \alpha\right) = -\sqrt{2} \sin \alpha.$

100. $\cos(\theta + \phi) + \sin(\theta - \phi) = 2 \cos\left(\dfrac{\pi}{4} - \theta\right) \cos\left(\dfrac{\pi}{4} + \phi\right).$

101. $2 \sin\left(\alpha + \dfrac{\pi}{4}\right) \sin\left(\alpha - \dfrac{\pi}{4}\right) = \sin^2 \alpha - \cos^2 \alpha.$

102. $\sin\left(\dfrac{\pi}{4} + \alpha\right) - \sin\left(\dfrac{\pi}{4} - \alpha\right) = \sqrt{2} \sin \alpha.$

103. $\sin 40^\circ - \sin 10^\circ = \dfrac{\sqrt{3} - 1}{\sqrt{2}} \cos 25^\circ.$

104. $\cos 3x - \cos x = -2 \sin^2 x \cos x.$ **105.** $\dfrac{\sin 75^\circ + \sin 15^\circ}{\sin 75^\circ - \sin 15^\circ} = \sqrt{3}.$

106. $\dfrac{\cos x + \cos y}{\cos x - \cos y} = -\cot \dfrac{x + y}{2} \cot \dfrac{x - y}{2}.$

107. $\dfrac{\sin 50^\circ + \sin 40^\circ}{\cos 50^\circ + \cos 40^\circ} = 1.$ **108.** $\dfrac{\sin 140^\circ + \sin 80^\circ}{\sin 140^\circ - \sin 80^\circ} = -\sqrt{3} \cot 20^\circ.$

109. $\dfrac{(\sin \alpha + \sin \beta)(\cos \alpha + \cos \beta)}{(\sin \alpha - \sin \beta)(\cos \alpha - \cos \beta)} = -\cot^2 \dfrac{\alpha - \beta}{2}.$

110. $\dfrac{(\sin \alpha + \sin \beta)(\cos \alpha - \cos \beta)}{(\sin \alpha - \sin \beta)(\cos \alpha + \cos \beta)} = -\tan^2 \dfrac{\alpha + \beta}{2}.$

111. $\dfrac{(\sin 75^\circ + \sin 15^\circ)(\cos 75^\circ + \cos 15^\circ)}{(\sin 75^\circ - \sin 15^\circ)(\cos 75^\circ - \cos 15^\circ)} = -3.$

112. $\dfrac{\cos 2x + \cos 12x}{\cos 6x + \cos 8x} + \dfrac{\cos 7x - \cos 3x}{\cos x - \cos 3x} + \dfrac{2 \sin 4x}{\sin 2x} = 0.$

113. $\sin x + \sin 2x + \sin 3x = 4 \cos \tfrac{1}{2} x \cos x \sin \tfrac{3}{2} x.$

(*Hint.* Replace $\sin x + \sin 3x$ by $2 \sin 2x \cos x$ and $\sin 2x$ by $2 \sin x \cos x$; from these results factor out $2 \cos x$ and combine the remainders by the formula for $\sin u + \sin v$.)

114. $\sin x - \sin 2x + \sin 3x = 4 \sin \tfrac{1}{2} x \cos x \cos \tfrac{3}{2} x.$

115. $\cos x - \cos 2x + \cos 3x = 1 - 4 \sin \tfrac{1}{2} x \cos x \sin \tfrac{3}{2} x.$

116. $\dfrac{\sin \theta + \sin 2\theta + \sin 3\theta}{\cos \theta + \cos 2\theta + \cos 3\theta} = \tan 2\theta.$

117. $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0.$

118. $\cos \theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 4 \cos \theta \cos 2\theta \cos 4\theta.$

119. $\sin \theta + \sin 3\theta + \sin 5\theta + \sin 7\theta = 16 \sin \theta \cos^2 \theta \cos^2 2\theta.$

120. $4 \sin^2 \phi \cos^2 \phi + (\cos^2 \phi - \sin^2 \phi)^2 = 1.$

121. $(\cos x \cos y + \sin x \sin y)^2 + (\sin x \cos y - \cos x \sin y)^2 = 1.$

122. $\dfrac{\tan 3x - \tan x}{1 + \tan 3x \tan x} = \tan 2x.$

123. $\dfrac{\tan (n + 1)\theta - \tan n\theta}{1 + \tan (n + 1)\theta \tan n\theta} = \tan \theta.$

124. $\dfrac{\tan(\theta+\phi)-\tan\phi}{1+\tan(\theta+\phi)\tan\phi}=\tan\theta.$

125. $\dfrac{\tan(\theta-\phi)+\tan\phi}{1-\tan(\theta-\phi)\tan\phi}=\tan\theta.$

126. $\sin n\theta\cos\theta+\cos n\theta\sin\theta=\sin(n+1)\theta.$

127. $2\csc 4x+2\cot 4x=\cot x-\tan x.$

128. $\dfrac{1-\cos 3x}{1-\cos x}=(1+2\cos x)^2.$ **129.** $\dfrac{\sin 3\theta}{\sin\theta}-\dfrac{\cos 3\theta}{\cos\theta}=2.$

130. If $\tan x=\dfrac{b}{a}$, show that $\sqrt{\dfrac{a+b}{a-b}}+\sqrt{\dfrac{a-b}{a+b}}=\dfrac{2\cos x}{\sqrt{\cos 2x}}.$

131. $4\cos^3 x\sin 3x+4\sin^3 x\cos 3x=3\sin 4x.$

132. $\sin^3 x+\sin^3(120^\circ+x)+\sin^3(240^\circ+x)=-\frac{3}{4}\sin 3x.$

133. $\cos 6x=16(\cos^6 x-\sin^6 x)-15\cos 2x.$

134. $1+\tan^6 x=\sec^4 x(\sec^2 x-3\sin^2 x).$

135. $\dfrac{3\sin x-\sin 3x}{3\cos x+\cos 3x}=\tan^3 x.$

136. $\sin 2x\sin 2y=\sin^2(x+y)-\sin^2(x-y).$

137. $\sin 5\alpha\sin\alpha=\sin^2 3\alpha-\sin^2 2\alpha.$

138. $\cos^4\alpha=\frac{1}{8}(3+4\cos 2\alpha+\cos 4\alpha).$

139. $\cos 2x+\cos 2y+\cos 2z+\cos 2(x+y+z)=4\cos(x+y)\cos(y+z)$
$\cos(z+x).$

140. $\sin^2 x+\sin^2 y+\sin^2 z+\sin^2(x+y+z)=2-2\cos(x+y)\cos(y+z)$
$\cos(z+x).$

141. $\cos^2 x+\cos^2 y+\cos^2 z+\cos^2(x+y-z)=2+2\cos(x+y)\cos(x-z)$
$\cos(y-z).$

142. $\sin(x-y-z)-\sin x-\sin y-\sin z=4\sin\dfrac{x-y}{2}\sin\dfrac{x-z}{2}\sin\dfrac{y+z}{2}.$

143. $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=\sin 2(\alpha+\beta+\gamma)+4\sin(\alpha+\beta)\sin(\beta+\gamma)$
$\sin(\alpha+\gamma).$

144. $\sin(\alpha+\beta-\gamma)+\sin(\alpha-\beta+\gamma)+\sin(\beta+\gamma-\alpha)-\sin(\alpha+\beta+\gamma)$
$=4\sin\alpha\sin\beta\sin\gamma.$

145. $\cos(\alpha+\beta-\gamma)+\cos(\beta+\gamma-\alpha)+\cos(\alpha+\gamma-\beta)+\cos(\alpha+\beta+\gamma)$
$=4\cos\alpha\cos\beta\cos\gamma.$

146. Show that the equation $\sin x=a+\dfrac{1}{a}$ is impossible.

147. For what values of a will the equation $2\cos x=a+\dfrac{1}{a}$ give possible values for x?

148. Show that $2\sin\dfrac{x}{2}=-\sqrt{1+\sin x}+\sqrt{1-\sin x}$, when x lies between 180° and 360°.

149. Show that $2\cos\dfrac{x}{2}=-\sqrt{1+\sin x}-\sqrt{1-\sin x}$, when x lies between 180° and 360° degrees.

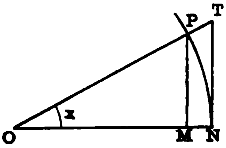
150. When x lies between 270° and 360°, show that

$$2\sin\frac{x}{2}=\sqrt{1-\sin x}-\sqrt{1+\sin x}.$$

CHAPTER IV

SMALL ANGLES. INVERSE FUNCTIONS. TRIGONOMETRIC EQUATIONS

49. The limits of the ratios $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$. Let $x = \angle NOP$ (figure) lie between 0° and 90°; let NP be a circular arc with center at O, and MP and $NT \perp ON$. Then



$$MP < NP < NT;$$

hence $$\frac{MP}{OP} < \frac{NP}{OP} < \frac{NT}{OP},$$

or $$\sin x < x \text{ (radians)} < \tan x.$$

That is, *the radian measure of any acute angle lies between the sine and the tangent of the angle.*

From the last inequality we have, on dividing by $\sin x$,

$$1 < \frac{x}{\sin x} < \sec x.$$

Suppose x to decrease and approach 0. Then $\sec x \doteq 1$, and consequently also $\frac{x}{\sin x} \doteq 1$ and $\frac{\sin x}{x} \doteq 1$.

Hence $$\lim_{x \doteq 0} \frac{\sin x}{x} = 1.$$

Dividing the third of the above inequalities by $\tan x$, we have

$$\cos x < \frac{x}{\tan x} < 1;$$

letting x approach zero we have

$$\lim_{x \doteq 0} \frac{\tan x}{x} = 1.$$

Hence, *the ratio of either the sine or the tangent to the angle (in radians) approaches* 1 *as its limit when the angle approaches zero.*

When angle x is small, these ratios will be nearly equal to 1 that is,

$$\frac{\sin x}{x} = 1 + e \quad \text{and} \quad \frac{\tan x}{x} = 1 + e_1,$$

where e and e_1 are small quantities. Hence

$$\sin x = x + ex \quad \text{and} \quad \tan x = x + e_1 x.$$

Neglecting the small terms ex and $e_1 x$, we have

$$\sin x = \tan x = x \textit{ approximately}, \text{ when } x \text{ is small.}$$

Hence *when x is small, $\sin x$ and $\tan x$ are nearly equal to x (in radians).*

The degree of this approximation is indicated by the following values:

Angle x			
Degrees	radians	$\sin x$	$\tan x$
1°	.0174532925+	.0174524064+	.0174550649+
1′	.0002908882+	.0002908882+	.0002908882+
1″	.0000048481+	.0000048481+	.0000048481+

Exercises.

1. How large may x be if the approximations

$$\sin x = x \quad \text{and} \quad \tan x = x$$

are to be correct to four places inclusive? (Table.)

2. In what decimal place is the error of the *approximations*

$$\sin 30° = 30 \sin 1° \quad \text{and} \quad \tan 30° = 30 \tan 1°?$$

3. How large may n be if the approximations

$$\sin n° = n \sin 1° \quad \text{and} \quad \tan n° = n \tan 1°$$

are to be correct to three decimals inclusive?

4. As in exercise 3, for the approximations

$$\sin n' = n \sin 1' \quad \text{and} \quad \tan n' = n \tan 1'.$$

5. If the diameter of a ball is 1 ft., show that its *angular diameter* at a distance of 100 ft. from the eye will be nearly .01 radians. If diameter is d and distance D, show that its angular diameter will be nearly $d \div D$ radians, provided D is large compared with d.

6. Apply the last result to Exercises 15, 16, 17, 18 of (**23**).

50. Inverse Trigonometric Functions. — It is often convenient to specify an angle, not by its degree or radian measure, but by the value of one of its functions. Thus we may speak of 30° as "an angle whose sine is $\frac{1}{2}$." There is of course an ambiguity here, since 30° is only one of the angles whose sine is $\frac{1}{2}$.

If x is an angle whose sine is a, we write

$$x = \sin^{-1} a,$$

which may be read "*x equals an angle whose sine is a,*" or "*x equals the inverse sine of a,*" or "*x equals anti-sine a.*"

Similarly the equation

$$x = \tan^{-1} a$$

is read "*x equals an angle whose tangent is a,*" or "*x equals the inverse tangent of a,*" or "*x equals anti-tangent a,*" and so on for the other functions.

Obviously the equations

$$x = \sin^{-1} a \quad \text{and} \quad \sin x = a$$

are equivalent. Similarly for

$$x = \tan^{-1} a \quad \text{and} \quad \tan x = a,$$
$$x = \sec^{-1} a \quad \text{and} \quad \sec x = a,$$

and so on.

It should be noted that "$^{-1}$" in $\sin^{-1} a$ is *not an exponent;* it might equally well have been written as a subscript, $\sin_{-1} x$, or in any other convenient way. The reason for writing it as above will appear by noting that, according to the laws of exponents, the algebraic equations

$$x = b^{-1} a \quad \text{and} \quad bx = a$$

are equivalent.

When it is necessary to write $\sin x$ with an exponent -1, it should be written $(\sin x)^{-1}$, *not* $\sin^{-1} x$.

To avoid any suggestion of a negative exponent, the symbols *arc sin x*, *arc tan x*, etc., are often used in place of *$\sin^{-1} x$*, *$\tan^{-1} x$*, etc.

The smallest positive angle whose sine is a is often called the *principal value* of the symbol $\sin^{-1} a$. Similarly for the other functions.

Usage here is not uniform. Sometimes the *principal value* is taken as that angle whose terminal line lies nearest to the initial line. Thus $\tan^{-1}(-1)$ would be taken as $-45°$, instead of $+135°$.

51. Equations Involving Inverse Functions. — In this article we shall restrict the symbol for the inverse functions to mean *only the principal value* of the function. Thus, $\sin^{-1}\frac{1}{2}$ shall mean the angle 30° only, $\tan^{-1} 1 = 45°$, and so on.

Example 1. Show that $\sin^{-1}\frac{3}{5} = \cos^{-1}\frac{4}{5}$.

Let $\quad x = \sin^{-1}\frac{3}{5}$ and $y = \cos^{-1}\frac{4}{5}$;

to prove that $\quad x = y$,

or that $\quad \sin x = \sin y$.

(We use the sine for convenience; any other function might be used.)

Now $\quad \sin x = \frac{3}{5}$ since $x = \sin^{-1}\frac{3}{5}$.

Also $\quad \cos y = \frac{4}{5}$; hence $\sin y = \sqrt{1 - \cos^2 y} = \frac{3}{5}$. q.e.d.

Example 2. Show that $2\tan^{-1} 2 = \pi - \sin^{-1}\frac{4}{5}$.

Let $\quad x = \tan^{-1} 2$ and $y = \sin^{-1}\frac{4}{5}$;

to prove that $\quad 2x = \pi - y$,

or that $\quad \sin 2x = \sin(\pi - y) = \sin y$.

Now $\quad \sin 2x = 2\sin x \cos x$.

But $\quad \tan x = 2$; hence $\sin x = \frac{2}{\sqrt{5}}$ and $\cos x = \frac{1}{\sqrt{5}}$. (28)

Therefore $\quad \sin 2x = \frac{4}{5} = \sin y$. q. e. d.

Observe that if x were not restricted to be the principal value of $\tan^{-1} 2$, we might have $\sin x = -\frac{2}{\sqrt{5}}$.

Example 3. Show that $\tan^{-1}\frac{2}{3} + \tan^{-1} 2 + \tan^{-1} 8 = \pi$.

Let $\quad x = \tan^{-1}\frac{2}{3}$; $y = \tan^{-1} 2$; $z = \tan^{-1} 8$;

then $\quad \tan x = \frac{2}{3}$; $\tan y = 2$; $\tan z = 8$.

To prove that $\quad x + y + z = \pi$,

or that $\quad x + y = \pi - z$,

or that $\quad \tan(x + y) = \tan(\pi - z) = -\tan z$.

Now $\quad \tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{2}{3} + 2}{1 - \frac{4}{3}} = -8 = -\tan z$. q.e.d.

Example 4. Show that $\tan^{-1} a = \sin^{-1} \dfrac{a}{\sqrt{1+a^2}}$ when $a > 0$.

Let $x = \tan^{-1} a$ and $y = \sin^{-1} \dfrac{a}{\sqrt{1+a^2}}$;

then $\tan x = a$ and $\sin y = \dfrac{a}{\sqrt{1+a^2}}$.

To prove that $x = y$,
or that $\sin x = \sin y$.

Now since x and y stand for principal values, and a is positive, both angles are in the first quadrant.

Then from $\tan x = a$ we find (**28**)

$$\sin x = \frac{a}{\sqrt{1+a^2}},$$

which is $\sin y$. q.e.d.

Discuss the above example when the symbol for the inverse functions is assumed to stand for all angles having the function in question, instead of the principal value only.

52. Exercises.

1. Show that the equation in example 4 is not true for principal values when a is negative. (Try $a = -1$.)

Prove the following:

2. $\tan^{-1} \dfrac{4}{5} + \tan^{-1} \dfrac{1}{9} = \dfrac{\pi}{4}$.

3. $2 \tan^{-1} \frac{2}{3} = \tan^{-1} \frac{12}{5}$.

4. $\tan^{-1} 2 + \dfrac{\pi}{4} = \tan^{-1} (-3)$.

5. $\cot^{-1} 2 + \csc^{-1} \sqrt{10} = \dfrac{\pi}{4}$.

6. $\sin^{-1} \sqrt{\dfrac{3}{2}} + 2 \cos^{-1} \sqrt{\dfrac{3}{2}} = 120°$.

7. $2 \tan^{-1} 4 = \pi - \sin^{-1} \frac{8}{17}$.

8. $3 \sin^{-1} \frac{1}{4} = \sin^{-1} \frac{11}{16}$.

9. $2 \cot^{-1} 2 = \sec^{-1} \frac{5}{3}$.

10. $4 \tan^{-1} \dfrac{1}{5} = \tan^{-1} \dfrac{1}{239} + \dfrac{\pi}{4}$.

11. $\tan^{-1} \frac{11}{16} + \tan^{-1} \frac{2}{5} + \tan^{-1} (-\frac{3}{2}) = \pi$.

12. $\sin^{-1} \dfrac{4}{5} + \sin^{-1} \dfrac{8}{17} + \sin^{-1} \dfrac{13}{85} = \dfrac{\pi}{2}$.

13. $\cos^{-1} \frac{63}{65} + 2 \tan^{-1} \frac{1}{5} = \sin^{-1} \frac{3}{5}$.

14. $2 \tan^{-1} \frac{2}{3} - \csc^{-1} \frac{5}{3} = \sin^{-1} \frac{33}{65}$.

15. $\sin^{-1} a = \cos^{-1} \sqrt{1 - a^2}$, if $a > 0$.

16. $2 \tan^{-1} m = \tan^{-1} \dfrac{2m}{1 - m^2}$.

17. $2 \tan^{-1} (\cos 2\theta) = \tan^{-1} \left(\dfrac{\cot^2 \theta - \tan^2 \theta}{2} \right)$

Note. The equation of Exercise 10 was used to calculate the value of π to 707 places.

53. Trigonometric Equations. — A trigonometric equation is an equation which involves one or more trigonometric functions of one or more angles. Thus:

$$\sin^2 x + \cos x = 1;\ \tan\theta + \sec\theta = 3;\ \cot\alpha\csc\alpha = 2.$$

To find the values of the angle which satisfy such an equation, it is usually best to use a method adapted to the case in hand. We give here one general rule, which covers a considerable variety of cases.

Rule: *To solve a trigonometric equation, express all its terms by means of a single function; solve as an algebraic equation, considering this function as unknown; find the angles corresponding to the values of the function so obtained. Check all answers by substitution.*

Examples.

1. $\sin^2 x + \cos x = 1.$

Expressing all terms by means of $\cos x$, we have

$$1 - \cos^2 x + \cos x = 1, \quad \text{or} \quad \cos^2 x - \cos x = 0.$$

$$\therefore \quad \cos x = 0, \quad \text{or} \quad \cos x = 1.$$

Hence x may be any odd multiple of $\frac{\pi}{2}$ or any multiple of 2π; i.e., if n be any integer or zero,

$$x = \pm(2n+1)\frac{\pi}{2} \quad \text{or} \quad x = \pm 2n\pi.$$

Exercise. Check these answers by substitution.

2. $\tan\theta + \sec\theta = 3.$

Expressing all terms by means of $\tan\theta$, we have

$$\tan\theta \pm \sqrt{1+\tan^2\theta} = 3, \text{ or } \pm\sqrt{1+\tan^2\theta} = 3 - \tan\theta.$$

Squaring and reducing,

$$\tan\theta = \frac{4}{3}; \text{ hence } \quad \theta = 53^\circ 8' \pm n\pi.$$

When n is odd, these values of θ do not satisfy the given equation. Hence the solutions are

$$\theta = 53^\circ 8' \pm 2n\pi.$$

3. $\cot\alpha\csc\alpha = 2.$

Then $\qquad \pm\cot\alpha\sqrt{1+\cot^2\alpha} = 2, \quad \text{or} \quad \cot^4\alpha + \cot^2\alpha = 4.$

Hence $\qquad \cot\alpha = \pm\sqrt{-\frac{1}{2} \pm \frac{1}{2}\sqrt{17}}.$

Using the upper sign under the radical (the lower sign makes α imaginary), we have

$$\cot\alpha = \pm 1.2496+; \quad \text{hence} \quad \alpha = \pm 38^\circ 40' \pm n\pi.$$

When n is odd, the values of α must be discarded. Hence

$$\alpha = \pm\ 38^\circ\, 40' \pm 2\, n\pi.$$

The reason for the additional values in the last two examples is that in example 2 we really solved both the equations $\tan\theta \pm \sec\theta = 3$, and in example 3, both the equations $\cot\alpha \csc\alpha = \pm\ 2$.

Examples Illustrating Special Methods. — These depend chiefly on transforming the given equation by means of some of the standard formulas.

4. $2\sin^2 x - 3\sin x\cos x = 1$.

Since $2\sin^2 x = 1 - \cos 2x$ and $2\sin x \cos x = \sin 2x$, we have

$$1 - \cos 2x - \frac{3}{2}\sin 2x = 1, \quad\text{or}\quad \tan 2x = -\frac{2}{3}.$$

Hence $$2x = \tan^{-1}\left(-\frac{2}{3}\right) = -\ 33^\circ\ 41' \pm n\pi.$$

$$\therefore\quad x = -\ 16^\circ\ 50'.5 \pm n\frac{\pi}{2}.$$

Exercise. Check these answers. Solve the given equation by expressing $\cos x$ in terms of $\sin x$.

5. $\sin 3y - \sin 2y = 0$.

By formula (24) of (47) this becomes

$$2\cos\tfrac{5}{2}y\sin\tfrac{1}{2}y = 0.$$

Hence $$\cos\tfrac{5}{2}y = 0 \quad\text{or}\quad \sin\tfrac{1}{2}y = 0.$$

$$\frac{5}{2}y = \pm\ (2n+1)\frac{\pi}{2}, \quad\text{or}\quad \frac{1}{2}y = \pm\ n\pi.$$

$$\therefore\quad y = \pm\ (2n+1)\frac{\pi}{5}, \quad\text{or}\quad y = \pm\ 2n\pi.$$

6. $\cos x + \cos 3x + \cos 5x = 0$.

Since $\cos x + \cos 5x = 2\cos 3x\cos 2x$, we have

$$2\cos 3x\cos 2x + \cos 3x = 0, \quad\text{or}\quad \cos 3x\,(2\cos 2x + 1) = 0.$$

Hence $$\cos 3x = 0, \quad\text{or}\quad \cos 2x = -\frac{1}{2}.$$

$$3x = \pm\ (2n+1)\frac{\pi}{2}, \quad\text{or}\quad 2x = \pm\ \frac{2\pi}{3} \pm 2n\pi.$$

$$\therefore\quad x = \pm\ (2n+1)\frac{\pi}{6} \quad\text{or}\quad \pm\frac{\pi}{3} \pm n\pi.$$

7. $\tan 4\alpha\tan 5\alpha = 1$.

This may be written $\tan 4\alpha = \cot 5\alpha$. But when the tangent of an angle A equals the cotangent of an angle B, $A + B$ must be an odd multiple of $\frac{\pi}{2}$.

Hence $$4\alpha + 5\alpha = \pm\ (2n+1)\frac{\pi}{2}$$

$$\therefore\quad \alpha = \pm\ (2n+1)\frac{\pi}{18}$$

Here α is any odd multiple of 10°.

Otherwise thus: $\tan 4\alpha - \cot 5\alpha = 0;$

hence
$$\frac{\sin 4\alpha}{\cos 4\alpha} - \frac{\cos 5\alpha}{\sin 5\alpha} = 0;$$

or
$$\frac{\sin 4\alpha \sin 5\alpha - \cos 4\alpha \cos 5\alpha}{\cos 4\alpha \sin 5\alpha} = -\frac{\cos 9\alpha}{\cos 4\alpha \sin 5\alpha} = 0.$$

$$\therefore \quad \cos 9\alpha = 0, \quad \text{or} \quad 9\alpha = \pm(2n+1)\frac{\pi}{2}.$$

Exercise 1. Check these answers. Draw figures for several values of α as 10°, 30°, 50°, 70°. Discuss the case $\alpha = 90°$.

Exercise 2. In example 7, in passing from the first equation to the second we divide by $\tan 5\alpha$, which is permissible only if $\tan 5\alpha \neq 0$. Justify the division.

Exercise 3. Justify the division by $\cos 2x$ in example 4.

8. $a \sin\theta + b\cos\theta = c.$

We might reduce to $\sin\theta$ or $\cos\theta$ and proceed according to the rule of (**53**). A method much preferred in practice is as follows.

In place of a and b introduce two new constants m and M such that

$$\begin{cases} a = m\cos M, \\ b = m\sin M; \end{cases} \quad \text{whence} \quad \begin{cases} m = \sqrt{a^2+b^2} \\ M = \tan^{-1}\dfrac{b}{a}. \end{cases}$$

The given equation then becomes

$$m(\sin\theta\cos M + \cos\theta\sin M) = c \quad \text{or} \quad \sin(\theta + M) = \frac{c}{m}.$$

Hence if we let $\sin^{-1}x$ represent all angles whose sine is x,

$$\theta + M = \sin^{-1}\frac{c}{m}, \quad \text{or} \quad \theta = \sin^{-1}\frac{c}{m} - M.$$

$$\therefore \quad \theta = \sin^{-1}\frac{c}{\sqrt{a^2+b^2}} - \tan^{-1}\frac{b}{a}.$$

54. Graphic Solution. As an example, we take the equation

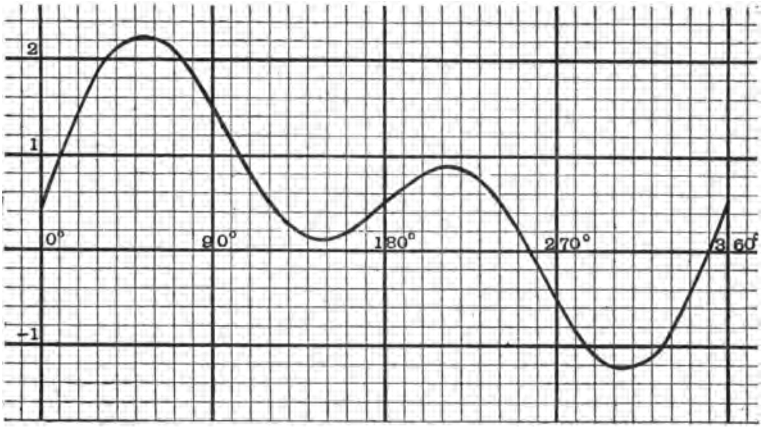
$$\sin 2\theta + \sin\theta + \tfrac{1}{2} = 0.$$

We want the values of θ which reduce the expression $\sin 2\theta + \sin\theta + \frac{1}{2}$ to zero.

Let
$$y = \sin 2\theta + \sin\theta + \tfrac{1}{2}.$$

Calculate y for a series of values of θ, as $\theta = 0°, 10°, 20°, \ldots,$ and plot the points (θ, y) in rectangular coördinates. The resulting curve will show the approximate values of θ for which y is zero. Any convenient scales may be used on the axes of θ and y.

Let the student read off the required solutions from the graph below.



Graph of the expression $\sin 2\theta + \sin\theta + \frac{1}{2}$.

Exercise. By means of this graph solve the equations

(a) $\sin 2\theta + \sin\theta = 0;$
(b) $\sin 2\theta + \sin\theta = 1;$
(c) $\sin 2\theta + \sin\theta = \frac{1}{2}.$

55. Exercises. Solve the following equations:

1. $2\sin^2 x - 3\cos x = 0.$
2. $4\sin^2\alpha + 1 = 8\cos\alpha.$
3. $\sin\alpha - \cos\alpha = \sqrt{2}.$
4. $\tan\theta - \cot\theta = 2.$
5. $\tan\beta + 3\cot\beta = 4.$
6. $3\sin^2 x + 4 = 7\sin x.$
7. $2(1 - \sin\theta) = \cos\theta.$
8. $15\sin\theta - 5\cos\theta = 1.$
9. $\cos 2x = \cos^2 x.$
10. $2\cos 2x = 1 + 2\sin x.$
11. $4\cot 2\theta = \cot^2\theta - \tan^2\theta.$
12. $\cos\theta = \sin 2\theta.$
13. $\tan 2x = 3\tan x.$
14. $\sin 2y = \cos 3y.$
15. $\tan\alpha = \cot 3\alpha.$
16. $\cot 8\phi = \tan\phi.$
17. $\sec px = \csc qx.$
18. $\tan y = \cot 6y.$
19. $\sin r\theta = \sin s\theta.$
20. $\cot(30° - x) = \tan(30° + 3x).$
21. $1 + \tan\beta = \tan\left(\frac{\pi}{4} + \beta\right).$
22. $\sin 4\alpha = \cos 5\alpha.$
23. $\sin(60° - x) - \sin(60° + x) = \frac{1}{2}\sqrt{3}.$
24. $\sin 2\theta + \sin 4\theta = \sqrt{2}\cos\theta.$
25. $\sin(30° + \theta) - \cos(60° + \theta) = -\frac{1}{2}\sqrt{3}.$
26. $\sin 4\alpha = \cos 3\alpha + \sin 2\alpha.$
27. $\sin 3\beta + \sin\beta = \cos\beta - \cos 3\beta.$
28. $\sin x + \sin 2x + \sin 3x = 0.$
29. $\sin x + \sin 3x + \sin 5x = 0.$
30. $\cos x + \cos 2x = \cos\frac{1}{2}x.$

Solve some of the above equations graphically, in particular, 1, 2, 4, 5, 7, 8, 12, 13, 14, 15, 26, 28, 29.

56. Simultaneous Trigonometric Equations. — We shall now give some examples to illustrate methods for solving a system of simultaneous trigonometric equations for several unknown quantities. To express answers concisely, we shall now use the symbols for the inverse functions to mean *all* the angles determined by the given function.

Examples.

1. Solve for r and θ:
$$r\cos\theta = x,$$
$$r\sin\theta = y.$$

Squaring and adding, $r^2 = x^2 + y^2$;

hence $r = \pm\sqrt{x^2+y^2}$.

Divide the second equation by the first,

$$\tan\theta = \frac{y}{x};\quad \text{hence}\quad \theta = \tan^{-1}\frac{y}{x}.$$

2. Solve for α and β:
$$a\sin\alpha + b\sin\beta = c,$$
$$d\sin\alpha + e\sin\beta = f.$$

Solve for $\sin\alpha$ and $\sin\beta$ as unknowns; hence get α and β.

Exercise. Carry out the solution of example 2. Is the solution possible for all values of $a, b, \ldots, f$?

3. Solve for r and θ:
$$ar\sin\theta + br\cos\theta = c,$$
$$a'r\sin\theta + b'r\cos\theta = c'.$$

Solve for $r\sin\theta$ and $r\cos\theta$ as unknown; then proceed as in example 1.

Exercise. Carry out the solution in example 3.

4. Solve for x and y:
$$y = \sin x,$$
$$y = \sin 2x.$$

Subtracting, $\sin 2x - \sin x = 0$ or $2\sin x\cos x - \sin x = 0$.

Hence $\sin x = 0$ or $\cos x = \frac{1}{2}$.

$$\therefore\quad x = \pm n\pi \quad\text{or}\quad \pm 60° \pm 2n\pi.$$
$$\therefore\quad y = 0 \quad\text{or}\quad \pm\tfrac{1}{2}\sqrt{3}.$$

Exercise. Solve example 4 graphically.

5. Solve for y and t:
$$y = a\sin(nt+b),$$
$$y = a'\sin(nt+b').$$

Equating the values of y, and expanding,

$$a(\sin nt\cos b + \cos nt\sin b) = a'(\sin nt\cos b' + \cos nt\sin b').$$

Dividing by $\cos nt$ and solving for $\tan nt$,

$$\tan nt = \frac{a'\sin b' - a\sin b}{a\cos b - a'\cos b'}.$$

This determines a set of values of nt. Then y is obtained by substituting in either of the given equations.

6. Solve for r, θ, and ϕ:

$$x = r\cos\theta\cos\phi,$$
$$y = r\cos\theta\sin\phi,$$
$$z = r\sin\theta.$$

Dividing the second equation by the first, we have

$$\frac{y}{x} = \tan\phi;\quad \text{hence}\quad \phi = \tan^{-1}\frac{y}{x}.$$

Squaring the first two equations and adding,

$$x^2 + y^2 = r^2\cos^2\theta;\quad \text{hence}\quad r\cos\theta = \pm\sqrt{x^2+y^2}.$$

Combining this result with the third equation, as in example 1, we have

$$\tan\theta = \frac{z}{\pm\sqrt{x^2+y^2}};\quad \text{hence}\quad \theta = \tan^{-1}\frac{z}{\pm\sqrt{x^2+y^2}}.$$
$$r^2 = x^2 + y^2 + z^2.$$

57. Exercises.

Solve for r and θ:

1. $r\cos\theta = 3,$
$r\sin\theta = 4.$

2. $r\cos\theta = 12,$
$r\sin\theta = -5.$

3. $r\cos\theta = -9,$
$r\sin\theta = -40.$

4. $r\cos\theta + 2r\sin\theta = 3,$
$r\sin\theta = 1.$

5. $r(2\sin\theta + 3\cos\theta) = 1,$
$r(\sin\theta + 4\cos\theta) = 1.$

6. $r\sin\left(\theta + \frac{\pi}{4}\right) = 2,$
$r\cos\left(\theta - \frac{\pi}{4}\right) = 1.$

7. $r = \sin\left(\theta + \frac{\pi}{4}\right),$
$2r = \sin\left(\theta - \frac{\pi}{4}\right).$

8. $r = 2\sin\left(2\theta + \frac{\pi}{3}\right),$
$r = 3\sin\left(\theta + \frac{2\pi}{3}\right).$

Suggestion. Put $x = \theta + \frac{2\pi}{3}$;
then $\quad 2\theta + \frac{\pi}{3} = 2x - \pi.$

Solve for r, θ, and ϕ:

9. $r\cos\theta\cos\phi = 3,$
$r\cos\theta\sin\phi = 4,$
$r\sin\theta = 5.$

10. $r\cos\theta\cos\phi = -1,$
$r\cos\theta\sin\phi = 1,$
$r\sin\theta = -2.$

Eliminate θ from the following pairs of equations:

11. $x = r\cos\theta;\quad y = r\sin\theta.$

12. $x = a\cos\theta;\quad y = b\sin\theta.$

13. $x = a^3\cos^3\theta;\quad y = b^3\sin^3\theta.$

14. $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1;\quad \frac{x}{a}\sin\theta - \frac{y}{b}\cos\theta = -1.$

15. Eliminate θ and ϕ from the equations
$x = r\cos\theta\cos\phi;\quad y = r\cos\theta\sin\phi;\quad z = r\sin\theta.$

16. Eliminate θ and ϕ from the equations
$x = a\cos\theta\cos\phi;\quad y = b\cos\theta\sin\phi;\quad z = c\sin\theta.$

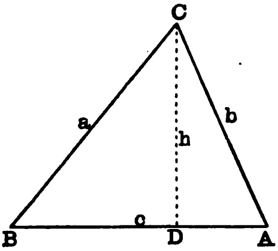
CHAPTER V

Oblique Plane Triangles

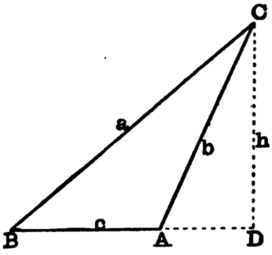
58. Between the six parts of a plane triangle there exist, aside from the angle-sum equal to 180°, two other fundamental relations which we proceed to obtain. Additional relations will then be derived from these.

The Law of Sines. — *In any plane triangle, the sides are proportional to the sines of the opposite angles.*

Let ABC be the triangle, CD one of its altitudes. Two cases arise, according as D falls within or without the base (figures).



First figure.



Second figure.

From $\triangle ACD$, $h = b \sin A$; $h = b \sin (\pi - A) = b \sin A$;

From $\triangle BCD$, $h = a \sin B$; $h = a \sin B$;

equating the values of h, we have in either case

$$b \sin A = a \sin B, \quad \text{or} \quad a : b = \sin A : \sin B.$$

By drawing perpendiculars from the other vertices and combining results we have the **Law of Sines**,

$$\textbf{(1)} \qquad a : b : c = \sin A : \sin B : \sin C.$$

59. The Law of Cosines. — *In any plane triangle, the square of any side equals the sum of the squares of the other two sides, minus twice their product by the cosine of their included angle.*

In the above figures let $\overline{AD} = m$. Then

	First figure.	*Second figure.*
in $\triangle ACD$,	$m = b\cos A;$	$m = b\cos(\pi - A) = -b\cos A;$
in $\triangle BCD$,	$a^2 = h^2 + (c-m)^2$	$a^2 = h^2 + (c+m)^2$
	$= h^2 + c^2 - 2cm + m^2$	$= h^2 + c^2 + 2cm + m^2$
	$= b^2 + c^2 - 2cm.$	$= b^2 + c^2 + 2cm.$

Replacing m by its value above, we have in either case,

(2) $\mathbf{a^2 = b^2 + c^2 - 2bc\cos A.}$

(2′) Similarly, $\mathbf{b^2 = a^2 + c^2 - 2ac\cos B.}$

(2″) $\mathbf{c^2 = a^2 + b^2 - 2ab\cos C.}$

60. The Law of Tangents. — *In any plane triangle, the difference of two sides is to their sum as the tangent of half the difference of the opposite angles is to the tangent of half their sum.*

From the law of sines we have,

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

By composition and division, and subsequent reduction we have,

$$\begin{aligned}\frac{a-b}{a+b} &= \frac{\sin A - \sin B}{\sin A + \sin B}\\ &= \frac{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}\\ &= \cot\tfrac{1}{2}(A+B)\tan\tfrac{1}{2}(A-B).\end{aligned}$$

That is,

(3) $$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}.$$

Similarly,

(3′) $$\frac{a-c}{a+c} = \frac{\tan\frac{1}{2}(A-C)}{\tan\frac{1}{2}(A+C)},$$

(3″) $$\frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}.$$

The symmetry of these formulas makes them easy to remember. In actual practice, they are used in slightly modified form. Thus the first of them is written,

$$\tan\tfrac{1}{2}(A-B) = \frac{a-b}{a+b}\tan\tfrac{1}{2}(A+B).$$

61. Functions of the Half-Angles. — When the three sides of a triangle are known, its angles are best calculated by the formulas now to be derived.

From the law of cosines we have,

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}.$$

In practice this formula is not convenient unless a, b, and c happen to be simple numbers. Now

$$\sin \tfrac{1}{2} A = \sqrt{\frac{1 - \cos A}{2}}. \quad \text{Why not } \pm \sqrt{\frac{1 - \cos A}{2}}?$$

But
$$\begin{aligned} 1 - \cos A &= 1 - \frac{b^2 + c^2 - a^2}{2\,bc} \\ &= \frac{2\,bc - b^2 - c^2 + a^2}{2\,bc} \\ &= \frac{a^2 - (b - c)^2}{2\,bc} \\ &= \frac{(a + b - c)\,(a - b + c)}{2\,bc}. \end{aligned}$$

Let $2s = a + b + c$, or $s = \tfrac{1}{2}(a + b + c)$.

Then $a + b - c = 2(s - c)$, and $a - b + c = 2(s - b)$.

Then
$$1 - \cos A = \frac{4(s - b)\,(s - c)}{2\,bc},$$

and

(4)
$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s - b)\,(s - c)}{bc}}.$$

Similarly,

(4′)
$$\sin \tfrac{1}{2} B = \sqrt{\frac{(s - a)\,(s - c)}{ac}}.$$

(4″)
$$\sin \tfrac{1}{2} C = \sqrt{\frac{(s - a)\,(s - b)}{ab}}.$$

Observe that the sides appearing explicitly under the radical *include* the angle to be calculated.

To obtain $\cos \tfrac{1}{2} A$, we have

$$\cos \tfrac{1}{2} A = \sqrt{\frac{1 + \cos A}{2}}.$$

But
$$
\begin{aligned}
1 + \cos A &= 1 + \frac{b^2 + c^2 - a^2}{2\,bc}\\
&= \frac{(b+c)^2 - a^2}{2\,bc}\\
&= \frac{(b+c+a)\,(b+c-a)}{2\,bc}\\
&= \frac{4\,s\,(s-a)}{2\,bc}.
\end{aligned}
$$

Hence

$$\cos\tfrac{1}{2}A = \sqrt{\frac{s\,(s-a)}{bc}}. \tag{5}$$

Similarly,

$$\cos\tfrac{1}{2}B = \sqrt{\frac{s\,(s-b)}{ac}}. \tag{5'}$$

$$\cos\tfrac{1}{2}C = \sqrt{\frac{s\,(s-c)}{ab}}. \tag{5''}$$

Dividing sine by cosine we have

$$\tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)\,(s-c)}{s\,(s-a)}}. \tag{6}$$

$$\tan\tfrac{1}{2}B = \sqrt{\frac{(s-a)\,(s-c)}{s\,(s-b)}}. \tag{6'}$$

$$\tan\tfrac{1}{2}C = \sqrt{\frac{(s-a)\,s-b)}{s\,(s-c)}}. \tag{6''}$$

If
$$r = \sqrt{\frac{(s-a)\,(s-b)\,(s-c)}{s}},$$
then

$$\tan\tfrac{1}{2}A = \frac{r}{s-a}, \tag{7}$$

$$\tan\tfrac{1}{2}B = \frac{r}{s-b}, \tag{7'}$$

$$\tan\tfrac{1}{2}C = \frac{r}{s-c}. \tag{7''}$$

All these formulas should be memorized in *verbal form*, so that a single statement contains all three formulas of any one set. Thus,

In Any Plane Triangle:

Law of Sines. — *Any two sides are proportional to the sines of their opposite angles.*

Law of Cosines. — *The square of any side equals the sum of the squares of the other two sides minus twice their product by the cosine of their included angle.*

Law of Tangents. — *The sum of two sides is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

Half Angles. — *The sine of half an angle equals the square root of the product of s minus each of the including sides divided by the product of those sides.* (*s equals half the sum of the sides.*)

— *The cosine of half an angle equals the square root of s times s minus the opposite side divided by the product of the including sides.*

— *The tangent of half an angle equals r divided by s minus the opposite side, where r equals the square root of the product of s minus each of the three sides divided by s.*

62. Solution of Plane Oblique Triangles. — A triangle is determined, except in such cases as will be specially mentioned, when three parts are given, of which one at least must be a side. The calculation of the other parts is called "solving the triangle."

Four cases arise, according to the nature of the given parts.

I. *Given one side and two angles.*
II. *Given two sides and their included angle.*
III. *Given two sides and an opposite angle.*
IV. *Given three sides.*

The method for treating each case will now be considered.

63. Case I. Given one side and two angles, as A, B, a. Formulas for finding the other parts, C, b, c.

$$C = 180° - (A + B).$$

From the law of sines,

$$b = a\frac{\sin B}{\sin A}; \quad c = a\frac{\sin C}{\sin A}.$$

Check. It is important to have a check on the accuracy of the calculated parts. For this purpose use any formula involving as many as possible of these parts.

In this case we use the law of tangents in the form:

$$(c+b)\tan\tfrac{1}{2}(C-B) = (c-b)\tan\tfrac{1}{2}(C+B).$$

Example. Given $A = 50°$, $B = 60°$, $a = 150$. To find C, b, and c. We first make a graphic solution by constructing a triangle having the given parts.

Graphic Solution. Calculate $C = 180° - (A + B)$. Lay off a line segment equal to a and at its extremities construct angles B and C, prolonging their free sides until they meet at A (figure). Scale off the lengths of b and c. The figure shows approximately

$$b = 170, \quad c = 180.$$

No solution is possible when $A + B > 180°$.

Solution by Natural Functions.

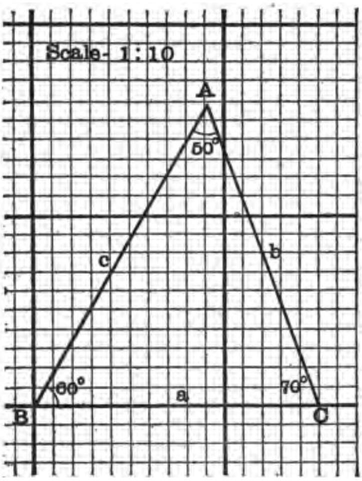
$$C = 180° - (50° + 60°) = 70°.$$

$$b = a\frac{\sin B}{\sin A} = \frac{150 \times .8660}{.7660}$$

$$= \mathbf{169.58.}$$

$$c = a\frac{\sin C}{\sin A} = \frac{150 \times .9397}{.7660}$$

$$= \mathbf{184.01.}$$



Check.

$$353.59 \times .0875 = 14.43 \times 2.1445,$$

or $\quad 30.94 = 30.94.$

Logarithmic Solution.

$$C = 180° - (A + B).$$

$$b = a\frac{\sin B}{\sin A}; \quad \log b = \log a + \log\sin B + \operatorname{colog}\sin A.$$

$$c = a\frac{\sin C}{\sin A}; \quad \log c = \log a + \log\sin C + \operatorname{colog}\sin A.$$

Check. $(c+b)\tan\tfrac{1}{2}(C-B) = (c-b)\tan\tfrac{1}{2}(C+B)$;

$$\log(c+b) + \log\tan\tfrac{1}{2}(C-B) = \log(c-b) + \log\tan\tfrac{1}{2}(C+B).$$

We now write out the following scheme, in which the entries are to be made in the order indicated by the numbers.

$$\begin{array}{lrlrl}
 & A + B = & (1) & C = 180^\circ - (A + B) = & (2) \\
 & \log a = & (3) & \log a = & (4) \\
 & \log \sin B = & (5) & \log \sin C = & (6) \\
 & \operatorname{colog} \sin A = & \underline{(7)} & \operatorname{colog} \sin A = & \underline{(8)} \\
 & \log b = & (9) & \log c = & (10) \\
 & b = & (11) & c = & (12) \\
\textit{Check.} & c + b = & (13) & c - b = & (14) \\
 & C - B = & (15) & C + B = & (16) \\
 & \log (c + b) = & (17) & \log (c - b) = & (18) \\
 & \log \tan \tfrac{1}{2} (C - B) = & \underline{(19)} & \log \tan \tfrac{1}{2} (C + B) = & \underline{(20)}
\end{array}$$

Now turn to the tables and take out all the logarithms required, inserting them in their proper places. Add to obtain log b and log c. Look up b and c and apply the check. If the sums in the check agree, or differ only slightly in the last figure, the numerical work is correct. Then look up b and c.

On making these calculations with the data in our example the scheme appears as below.

$$\begin{array}{lrlrl}
 & A + B = & 110^\circ. & C = 180^\circ - 110^\circ = & 70^\circ \\
 & \log a = & 2.1761 & \log a = & 2.1761 \\
 & \log \sin B = & 9.9375 & \log \sin C = & 9.9730 \\
 & \operatorname{colog} \sin A = & \underline{0.1157} & \operatorname{colog} \sin A = & \underline{0.1157} \\
 & \log b = & 2.2293 & \log c = & 2.2648 \\
 & b = & \mathbf{169.56} & c = & \mathbf{184.00} \\
\textit{Check.} & c + b = & 353.56 & c - b = & 14.44 \\
 & C - B = & 10^\circ & C + B = & 130^\circ \\
 & \log (c + b) = & 2.5486 & \log (c - b) = & 1.1596 \\
 & \log \tan \tfrac{1}{2} (C - B) = & \underline{8.9420} & \log \tan \tfrac{1}{2} (C + B) = & \underline{0.3313} \\
 & \textit{Check sums:} & 1.4906 & & 1.4909
\end{array}$$

Remark. In calculating with four-place logarithms, three significant figures of the resulting numbers are usually correct. The fourth figure should be retained, but may be one or more units in error. It is rarely worth while to retain more than four significant figures.

A similar remark applies to 5-, 6-, and 7-place tables.

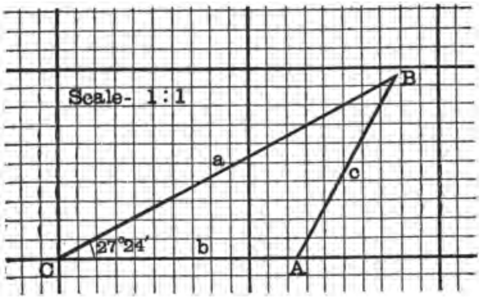
Exercises. Solve the following triangles, including graphic solutions.

1. $a = 1000$, $A = 50°$, $B = 75°$.
2. $a = 5.257$, $A = 62° 35'$, $B = 70° 43'$.
3. $b = 7.918$, $B = 77° 10'$, $C = 64° 50'$.
4. $c = 0.00835$, $B = 121° 35'$, $C = 35° 41'$.
5. $c = 3708$, $B = 59° 5'$, $C = 33° 15'$.

64. Case II. Given two sides and the included angle, as *a, b, C.*

Example. Given $b = 12.55$, $a = 20.63$, $C = 27° 24'$. Solve the triangle.

Graphic Solution. Construct angle C and on its sides lay off lengths a and b, starting from the vertex. Complete the triangle, and measure c, A, and B.



A solution is possible provided $0 < C < 180°$.

To solve the triangle we calculate $\frac{1}{2}(A + B)$ as the complement of $\frac{1}{2}C$; then $\frac{1}{2}(A - B)$ is calculated by formula (3). Angles A and B are then determined and hence all the angles are known. We can then compute c in two ways by means of the law of sines. The agreement of the two values of c furnishes a check on the computations.

Formulas.

$$\tfrac{1}{2}(A + B) = 90° - \tfrac{1}{2}C,$$

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} \tan \tfrac{1}{2}(A + B),$$

$$c = a\frac{\sin C}{\sin A} = b\frac{\sin C}{\sin B}. \qquad \textit{Check.}$$

Scheme for Logarithmic Solution.

$$\begin{array}{rl rl rl}
a = & (1) & \log(a-b) = & (6) & \frac{1}{2}(A+B) = & (5) \\
b = & \underline{(2)} & \operatorname{colog}(a+b) = & (7) & \frac{1}{2}(A-B) = & \underline{(14)} \\
a+b = & (3) & \log\tan\frac{1}{2}(A+B) = & \underline{(10)} & A = & (15) \\
a-b = & (4) & \log\tan\frac{1}{2}(A-B) = & (13) & B = & (16)
\end{array}$$

$$\begin{array}{l rl rl}
 & \log a = & (8) & \log b = & (9) \\
 & \log\sin C = & (11) & \log\sin C = & (12) \\
 & \operatorname{colog}\sin A = & \underline{(17)} & \operatorname{colog}\sin B = & \underline{(18)} \\
\textit{Check.} & \log c = & (19) & \log c = & (20) \\
 & c = & (21) & &
\end{array}$$

Logarithmic Solution.

$$\tfrac{1}{2}(A+B) = 90^\circ - \tfrac{1}{2}C = 90^\circ - 13^\circ\,42' = 76^\circ\,18'.$$

$$\begin{array}{rl rl rl}
a = & 20.63 & \log(a-b) = & 0.9074 & \frac{1}{2}(A+B) = & 76^\circ\,18' \\
b = & \underline{12.55} & \operatorname{colog}(a+b) = & 8.4792 & \frac{1}{2}(A-B) = & \underline{44^\circ\,58'.4} \\
a+b = & 33.18 & \log\tan\frac{1}{2}(A+B) = & \underline{0.6130} & A = & \mathbf{121^\circ\,16'.4} \\
a-b = & 8.08 & \log\tan\frac{1}{2}(A-B) = & 9.9996 & B = & \mathbf{31^\circ\,19'.6}
\end{array}$$

$$\begin{array}{l rl rl}
 & \log a = & 1.3145 & \log b = & 1.0986 \\
 & \log\sin C = & 9.6630 & \log\sin C = & 9.6630 \\
 & \operatorname{colog}\sin A = & \underline{0.0682} & \operatorname{colog}\sin B = & \underline{0.2841} \\
\textit{Check.} & \log c = & 1.0457 & \log c = & 1.0457 \\
 & c = & \mathbf{11.11} & &
\end{array}$$

Caution. Agreement of the two values of c is not a complete check; they may agree, yet both be wrong, due to an error in $\log \sin C$; check this very carefully.

Note. If side b were greater than side a, the difference $b - a$ would be negative, as also the difference $A - B$. To avoid negative differences in such cases, interchange letters in the formula for the law of tangents, and write it

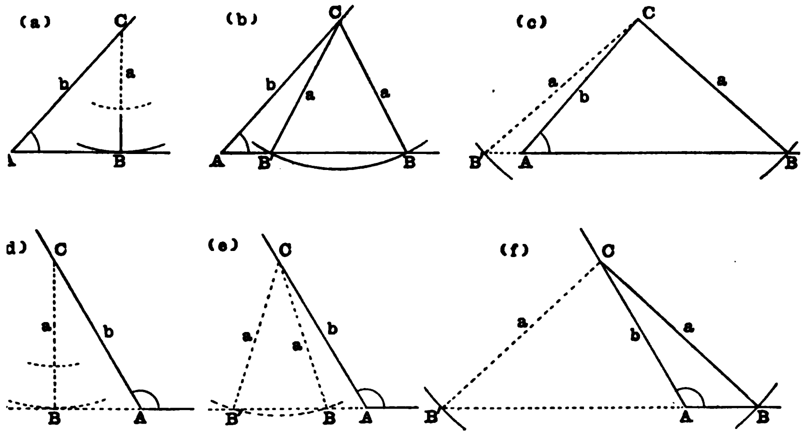
$$\tan\tfrac{1}{2}(B-A) = \frac{b-a}{b+a}\tan\tfrac{1}{2}(B+A).$$

Exercises. Solve the following triangles:

1. $a = 800$, $b = 895$, $C = 60^\circ$.
2. $a = 25.45$, $c = 21.60$, $B = 52^\circ\,30'$.
3. $a = 223$, $b = 402$, $C = 101^\circ\,40'$.
4. $b = 3124$, $c = 8976$, $A = 125^\circ\,32'$.
5. $b = .04544$, $c = .06400$, $A = 36^\circ\,08'$.

65. Case III. Given two sides and an opposite angle, as a, b, A.

This is known as the *ambiguous case.* We begin by studying the **Graphic Solution.** Lay off angle A and on one of its sides take $AC = b$. With C as center and radius equal to a, strike an arc of a circle. The figures show the various possibilities arising in the construction, the first three for $A < 90°$, the last three for $A > 90°$.



In each case the perpendicular from C on the other side of angle A is equal to $b \sin A$. Inspection of the figures then shows that

when $A < 90°$ and $a < b \sin A$, no triangle is possible;
when $A < 90°$ and $a = b \sin A$, a right triangle results;
when $A < 90°$ and $b > a > b \sin A$, two oblique triangles result;
when $A < 90°$ and $a \geqq b$, one oblique triangle results;
when $A > 90°$ and $a \leqq b$, no solution is possible;
when $A > 90°$ and $a > b$, one oblique triangle results.

It is always possible therefore to state in advance what the nature of the solution in a given case will be.

In a given numerical example the nature of the solution always becomes apparent during the progress of the computations.

Formulas. Given a, b, A.

$$\begin{cases} \sin B = \dfrac{b}{a}\sin A. \\ B' = 180^\circ - B. \end{cases} \quad \begin{cases} C = 180^\circ - (A + B). \\ C' = 180^\circ - (A + B'). \end{cases} \quad \begin{cases} c = a\dfrac{\sin C}{\sin A} = b\dfrac{\sin C}{\sin B}. \\ c' = a\dfrac{\sin C'}{\sin A} = b\dfrac{\sin C'}{\sin B'}. \end{cases}$$

Check. The agreement of the values of c and c' as calculated from the two expressions for each of them furnishes a partial check on the calculations. It does not guard against an error in log sin C, which may be checked independently. A surer check is furnished by the law of tangents.

In carrying out the calculations according to the formulas above, the various cases shown in the figures are indicated as follows:

(a) $\log \sin B \geqq 0$; no solution, or right triangle.

(b) retain both B and B'; two solutions.

(c) $A + B' > 180^\circ$, hence reject B'; one solution.

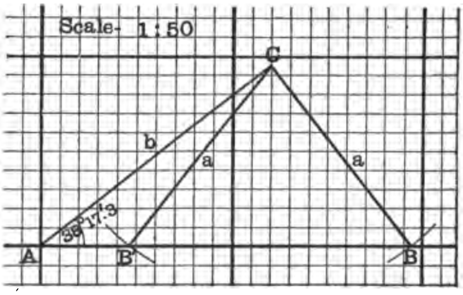
(d) $\log \sin B \geqq 0$; no solution.

(e) $A + B > 180^\circ$ and $A + B' > 180^\circ$; no solution.

(f) As in (c); one solution.

Example. Given $a = 602.3$, $b = 764.1$, $A = 38^\circ\ 17'.3$.

Graphic Solution. This is shown in the figure, from which the unknown parts may be scaled off.



Logarithmic Solution.

The fifth figure is carried to avoid accumulation of error. This is advisable if all possible accuracy is desired.

$$\begin{array}{rlrlrl}
\log b &= 2.88316 & \log a &= 2.77981 & \log b &= 2.88316\\
\operatorname{colog} a &= 7.22019 & \log\sin C &= 0.00000 & \log\sin C &= 0.00000\\
\log\sin A &= \underline{9.79217} & \operatorname{colog}\sin A &= \underline{0.20783} & \operatorname{colog}\sin B &= \underline{0.10448}\\
\log\sin B &= 9.89552 & \log c &= 2.98764 & \log c &= 2.98764\\
B &= \mathbf{51^\circ\,50'.0} & c &= \mathbf{971.9} & &\\
B' &= \mathbf{128^\circ\,10'.0} & \log a &= 2.77981 & \log b &= 2.88316\\
A+B &= 90^\circ\;\;7'.3 & \log\sin C' &= 9.36960 & \log\sin C' &= 9.36960\\
A+B' &= 166^\circ\,27'.3 & \operatorname{colog}\sin A &= \underline{0.20783} & \operatorname{colog}\sin B' &= \underline{0.10448}\\
C &= \mathbf{89^\circ\,52'.7} & \log c' &= 2.35724 & \log c' &= 2.35724\\
C' &= \mathbf{13^\circ\,32'.7} & c' &= \mathbf{227.6} & &
\end{array}$$

Exercises. Solve the triangles whose given parts are:

1. $a = 31.1$, $b = 37.4$, $A = 27^\circ\,18'$.
2. $a = .0878$, $b = .0972$, $A = 65^\circ\,20'$.
3. $a = 114.3$, $c = 134.6$, $A = 58^\circ\,6'.5$.
4. $b = 2.72$, $c = 5.56$, $B = 29^\circ\,55'$.
5. $b = 1392$, $c = 3218$, $C = 123^\circ\,39'$.

66. Case IV. Given the three sides, a, b, c.

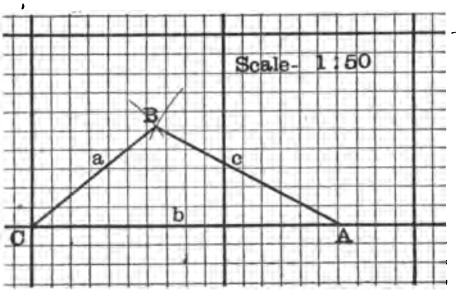
The angles may be calculated from either the sine, cosine, or tangent of the half-angles. When all three angles are wanted, it is best to use the tangent. There is no solution when one side equals or exceeds the sum of the other two.

Formulas. $s = \frac{1}{2}(a+b+c)$; $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$;

$$\tan\tfrac{1}{2}A = \frac{r}{s-a};\quad \tan\tfrac{1}{2}B = \frac{r}{s-b};\quad \tan\tfrac{1}{2}C = \frac{r}{s-c}.$$

Check. $\frac{1}{2}(A+B+C) = 90^\circ$.

Example. Given $a = 428.6$, $b = 806.2$, $c = 542.4$.



Graphic Solution. This is shown in the figure. By measuring we find $A = 29^\circ$, $B = 112^\circ$, $C = 38^\circ$.

Logarithmic Solution.

a	428.6	colog s	7.0513	$\frac{1}{2}A$	14° 47′.7
b	806.2	$\log(s-a)$	2.6628	$\frac{1}{2}B$	55° 51′.5
c	542.4	$\log(s-b)$	1.9159	$\frac{1}{2}C$	19° 20′.5
$2s$	1777.2	$\log(s-c)$	2.5393	Check	89° 59′.7
s	888.6		2|4.1693	A	**29° 35′.4**
$s-a$	460.0	$\log r$	2.0846	B	**111° 43′.0**
$s-b$	82.4	$\log\tan\frac{1}{2}A$	9.4218	C	**38° 41′.0**
$s-c$	346.2	$\log\tan\frac{1}{2}B$	0.1687		179° 59′.4
Check	1777.2	$\log\tan\frac{1}{2}C$	9.5453		

Exercises. Solve the triangles whose given parts are:

1. $a = 112,\quad b = 86\quad c = 98.$
2. $a = .6852,\quad b = .6284,\quad c = .6066.$
3. $a = 55.33,\quad b = 30.33,\quad c = 39.30.$
4. $a = .00150,\quad b = .00181,\quad c = .00294.$
5. $a = 1626,\quad b = 1448,\quad c = 3075.$

67. Areas of Oblique Plane Triangles. — Referring to the figures of (**58**), we see that h is the altitude drawn on side c as base. Hence if K denote the area of the triangle, we have

$$K = \tfrac{1}{2}\,hc = \tfrac{1}{2}\,ac\sin B. \qquad (h = a\sin B.) \tag{8}$$

Hence, *the area of a plane triangle equals half the product of two sides by the sine of their included angle.*

The area is also expressible in simple form in terms of the sides. In the formula above replace $\sin B$ by $2\sin\frac{1}{2}B\cos\frac{1}{2}B$. Then

$$\begin{aligned} K &= ac\sin\tfrac{1}{2}B\cos\tfrac{1}{2}B \\ &= ac\sqrt{\frac{(s-a)\,(s-c)}{ac}}\sqrt{\frac{s\,(s-b)}{ac}}, \end{aligned}$$

by (4′) and (5′) of (**61**). Hence,

$$K = \sqrt{s\,(s-a)(s-b)(s-c)} = rs. \tag{9}$$

When the given parts of the triangle are such that neither of the above formulas applies directly, it is usually best to calculate additional parts so that one of these formulas may be used.

68. Exercises and Problems.

1.
$b = 5818$,
$A = 36^\circ 56'$,
$B = 72^\circ 6'$.

2.
$a = 91.95$,
$b = 29.25$,
$c = 83.30$.

3.
$c = 1307$,
$A = 81^\circ 52'$,
$C = 55^\circ 41'$.

4.
$b = 167.10$,
$A = 65^\circ 49'.8$,
$B = 38^\circ 37'.4$.

5.
$a = .2018$,
$b = .1466$,
$C = 58^\circ 47'$.

6.
$b = 1032$,
$c = 1368$,
$A = 23^\circ 7'$.

7.
$a = 76.15$,
$b = 94.05$,
$A = 21^\circ 21'$.

8.
$a = 2748$,
$b = 8966$,
$A = 148^\circ 35'$.

9.
$a = .04353$,
$b = .00458$,
$c = .03951$,

10.
$b = 8310$,
$c = 6366$,
$C = 49^\circ 59'.7$.

11.
$a = 2.152$,
$c = 1.589$,
$A = 19^\circ 12'.7$.

12.
$a = 1064$,
$b = 1408$,
$C = 73^\circ$.

13.
$a = .1968$,
$c = .1183$,
$C = 22^\circ 32'$.

14.
$a = 3828$,
$b = 4146$,
$c = 2964$.

15.
$b = .00279$,
$c = .00233$,
$A = 57^\circ 53'$.

16.
$a = 2914$,
$c = 946$,
$B = 13^\circ 11'.7$.

17.
$a = .000598$,
$c = .000360$,
$A = 63^\circ 50'$.

18.
$b = 7265$,
$c = 3218$,
$C = 48^\circ 32'$.

19.
$b = .5064$,
$c = .7458$,
$C = 10^\circ 32'.8$.

20.
$a = 49.369$,
$b = 37.403$,
$c = 38.088$.

21.
$b = 64082$,
$A = 13^\circ 31'$,
$B = 15^\circ 9'.4$.

22.
$b = 3236$,
$c = 3610$,
$C = 56^\circ 34'.5$.

23.
$a = 0.01566$,
$c = 0.01307$,
$B = 42^\circ 27'$.

24.
$a = 3459$,
$B = 44^\circ 03'$,
$C = 67^\circ 10'$.

25.
$b = .1974$,
$B = 51^\circ 41'.8$,
$C = 93^\circ 46'.1$.

26.
$a = .0157$,
$b = .0428$,
$c = .0588$.

27.
$a = 385.2$,
$b = 455.3$,
$A = 41^\circ 13'$.

28.
$a = 165$,
$b = 345$,
$A = 69^\circ 18'$.

29.
$a = 632$,
$b = 741$,
$A = 27^\circ 18'$.

30.
$a = 10.33$,
$b = 5.03$,
$c = 6.68$.

31.
$b = 3110$,
$c = 1466$,
$A = 52^\circ 11'.2$.

32.
$a = 15.633$,
$b = 17.826$,
$c = 43.785$.

33.
$a = 11782$,
$b = 14216$,
$B = 50^\circ 20'.9$.

34.
$a = 44.44$,
$b = 77.78$,
$C = 58^\circ 49'$.

35.
$c = 0.03765$,
$A = 45^\circ 29'.5$,
$C = 120^\circ 15'$.

36.
$a = 10728$,
$c = 7574$,
$B = 104^\circ 20'$.

37.
$b = 97.16$,
$A = 21^\circ 13'.9$,
$B = 126^\circ 26'.4$.

38.
$a = 675$,
$c = 375$,
$A = 100^\circ 56'.7$.

39.
$a = .00932$,
$b = .00850$,
$B = 63^\circ 40'$.

40.
$a = .0762$,
$b = .0761$,
$B = 91^\circ 30'$.

In any triangle ABC, whose sides, opposite angles A, B, C, respectively, are a, b, c, show that:

41. $b(s-b)\cos^2\frac{A}{2} = a(s-a)\cos^2\frac{B}{2}.$

42. $a = b\cos C + c\cos B.$

43. $(a-b)(1+\cos C) = c(\cos B - \cos A).$

44. $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}.$

45. $(b+c-a)\tan\frac{A}{2} = (c+a-b)\tan\frac{B}{2}.$

46. $(b+c)(1-\cos A) = a(\cos B + \cos C).$

47. $(a^2-b^2+c^2)\tan B = (a^2+b^2-c^2)\tan C.$

48. $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$

49. The radius of the inscribed circle is $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$

50. The diameter of the circumscribed circle is $a \csc A$.

51. Find the lengths of diagonals and the area of a parallelogram two of whose sides are 5 ft. and 8 ft., their included angle being 60°.

52. Two sides of a parallelogram are a and b, their included angle C; show that the area is $ab \sin C$.

53. The sides of a triangle are in the ratio of 2 : 3 : 4; find the cosine of the smallest angle.

54. The angles of a triangle are as 1 : 2 : 3; the longest side is 100 ft.; solve the triangle.

55. The angles of a triangle are as 3 : 4 : 5; the shortest side is 500 ft.; solve the triangle.

56. The sides of a triangle are 4527, 7861, 6448; find the length of the median drawn to the shortest side. *Ans.* 6824.

57. In $\triangle ABC$, $a = 466$, $b = 572$, $c = 321$. Calculate the shortest altitude. *Ans.* 261.5.

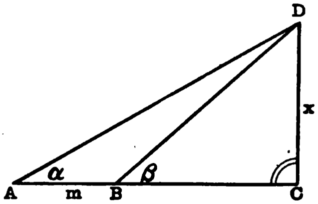
58. In $\triangle ABC$, $a = 336$, $b = 215$, $c = 252$. Calculate the length of the shortest median.

Exercises 59–90, which follow, are problems in "Heights and Distances," so-called; they indicate some of the applications of Trigonometry to mensuration.

For example, the figure of Exercise 59 is a general figure applying to such problems as are illustrated by Exercises 71 and 72 below. The figure of Exercise 70 applies to such problems as appear in Exercises 82 and 83, which represent actual observations of the flight of an aeroplane and of a meteor respectively.

In the figures, x, the unknown, is to be expressed in terms of the other parts, which are regarded as being given by measurement. Right angles are indicated by a double arc. In each case assume a set of numerical values for the given parts and calculate the numerical value of x.

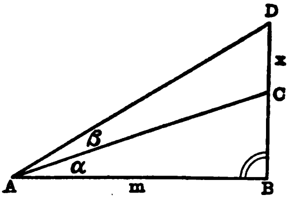
59.



$$x = m\frac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)}.$$

See (**32**).

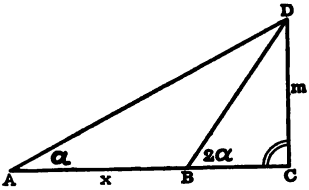
60.



$$x = m\frac{\sin\beta\sec\alpha}{\cos(\alpha+\beta)}.$$

($AC = m\sec\alpha$; $\angle ADC = 90° - (\alpha+\beta)$; then apply law of sines to $\triangle ACD$. Or, take $x = BD - BC$.)

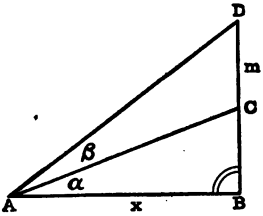
61.



$$x = m\csc 2\alpha.$$

(Note that $\triangle ABD$ is isosceles.)

62.



$$x = m\cos\alpha\cos(\alpha+\beta).$$

(First find AC in $\triangle ACD$.)

63. $x = BC + CD$,

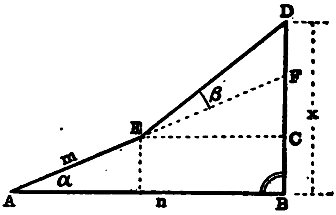
$BC = m\sin\alpha$,

$CD = (n - m\cos\alpha)\tan(\alpha+\beta)$;

or, $x = BF + FD$,

$BF = n\tan\alpha$,

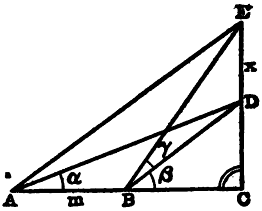
$$FD = (n\sec\alpha - m)\,\frac{\sin\beta}{\cos(\alpha+\beta)}.$$



64. $$x = m\frac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)}\left[\frac{\tan(\beta+\gamma)}{\tan\beta} - 1\right]$$

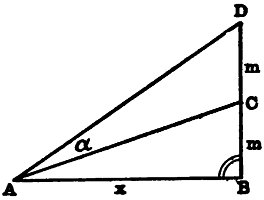
$$= m\frac{\sin\alpha\sin\gamma}{\sin(\beta-\alpha)\cos(\beta+\gamma)}.$$

(First find CD as in Ex. 59; then BC, then CE; then $x = CE - CD$. This gives first form; reduce to second form.)

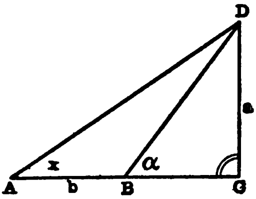


65. $x = \frac{m}{2}[\cot\alpha \pm \sqrt{\cot^2\alpha - 8}]$.

(Let $\angle BAC = \beta$; then $\tan\beta = \frac{m}{x}$, and $\tan(\alpha+\beta) = \frac{2m}{x}$; expand $\tan(\alpha+\beta)$; substitute value of $\tan\beta$, and solve for x.)



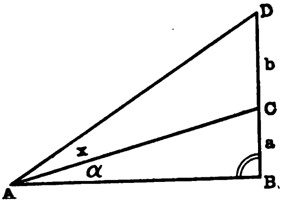
66.



$$x = \cot^{-1}\left(\frac{b}{a} + \tan\alpha\right).$$

$(\cot x = (AB + BC) \div a.)$

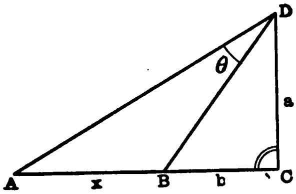
67.



$$x = \tan^{-1}\left(\frac{a+b}{a\cot\alpha}\right) - \alpha.$$

(Note that $\tan(x+\alpha) = (a+b) \div AB$.)

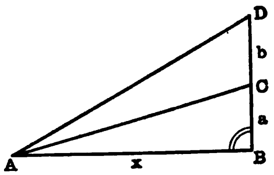
68.



$$x = a\cot\left(\tan^{-1}\frac{a}{b} - \theta\right) - b.$$

$(x + b = a\cot A;\ A = \tan^{-1}\frac{a}{b} - \theta.)$

69.



Given $\angle BAC = \angle CAD$.

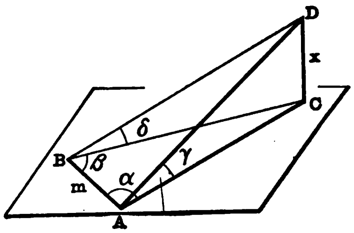
$$x = a\sqrt{\frac{b+a}{b-a}}.$$

70.

CD is $\perp$ to plane of $\triangle ABC$;
α and β are $\measuredangle$s of $\triangle ABC$;
γ and δ are $\measuredangle$s in vertical planes.

$$x = m\frac{\sin\beta\tan\gamma}{\sin(\alpha+\beta)};$$

or, $$x = m\frac{\sin\alpha\tan\delta}{\sin(\alpha+\beta)}.$$



71. From a level plain, the angle of elevation of a distant mountain top is 5° 50′; after approaching 4 miles, the angle is 8° 40′; how high is the mountain?

72. From a point on level ground the angle of elevation of the top of a hill is 14° 12′; on approaching 1000 ft., the angle is 17° 50′; how high is the hill? *Ans.* 1186 ft.

73. From level ground the angle of elevation of the top of a hill is 11° 30′; after approaching 3000 ft. up an incline of 3° 27′, the angle of elevation of the top is 21° 32′; how high is the hill?

74. From a point 60 ft. above sea level the angle between a distant ship and the sea horizon (the offing) is 20′; how far away is the ship? (Consider the surface of the sea as a plane, and the distance to the horizon 10 miles.) *Ans.* 8640 ft.

75. A tower 100 feet high has a mark 40 feet above the ground. How far from the foot of the tower will the two parts subtend equal angles?

76. A column 12 feet high stands on a pedestal 8 feet high. How far from the foot of the pedestal (and in the same horizontal plane with it) will column and pedestal subtend equal angles?

77. A flag-pole 30 feet high, standing on ground which slopes upward at an angle of 20°, casts a shadow 50 feet long and extending directly down the hill. What is the altitude of the sun?

78. The angle of elevation of the top of a building 100 ft. high is 60°; what will be the angle at double the distance?

79. From a station on level ground due south of a hill, the angle of elevation of the top is 15°; from a point 2000 ft. east of this station the angle of elevation is 12°; how high is the hill?

80. On level ground, 250 ft. from the foot of a building, the angles of elevation of the top and bottom of a flag-pole surmounting the building are 38° 43′ and 31° 2′ respectively; find the height of the building and the pole.

81. A flag-pole on a building subtends an angle of 7° 40′ at a point on the ground 100 ft. from the building; on approaching 20 ft., the pole subtends an angle of 7° 50′; find the height of the pole and the building.

82. To determine the height of an aeroplane, simultaneous observations from two stations were made as follows (see Ex. **70**): $m = 6236$ ft.; $\alpha = 72° 12'$, $\beta = 74° 10'$, $\gamma = 9° 24'$, $\delta = 9° 37'$. Show that the average of the two values of h is 1803 ft.

83. To determine the height of a meteor, simultaneous observations from two stations were made as follows (see Ex. **70**): $m = 18.3$ miles; $\alpha = 56° 35'$, $\beta = 104° 30'$, $\gamma = 53° 50'$, $\delta = 56° 45'$. Show that the average of the two values of h is 72.0 miles.

84. On approaching 1 mile toward a hill, the angle of elevation of its top is doubled; on approaching $\frac{2}{3}$ mile more, the angle is again doubled; how high is the hill? *Ans.* $\frac{1}{4}\sqrt{7}$ mi.

85. A building surmounted by a flag-pole 20 ft. high stands on level ground. From a point on the ground the angles of elevation of the top and the bottom of the pole are 53° 5′ and 45° 11′ respectively. How high is the building?

86. A and B are two points neither of which is visible from the other. To determine the distance AB, two stations C and D are chosen and the following measurements made: $CD = 500$ ft.; $\angle ACD = 30° 25' 15''$; $\angle ACB = 85° 40' 20''$; $\angle BDC = 35° 14' 50''$; $\angle BDA = 80° 20' 25''$; find AB. *Ans.* 969.2 ft.

87. In a chain of three non-overlapping triangles, the following data are known:

$$AB = 1000 \text{ ft.}$$

$$\begin{array}{lll} \triangle ABC, & \triangle ACD, & \triangle CDE, \\ \angle A = 44° 36', & \angle A = 56° 32', & \angle C = 55° 30', \\ \angle C = 40° 0'; & \angle C = 50° 20'; & \angle E = 77° 02'; \end{array}$$

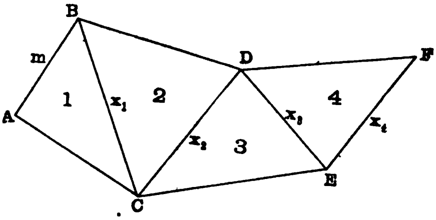
calculate DE. (Express DE in terms of AB and the necessary angles by the law of sines.)

88. In a chain of four non-overlapping triangles, the following data are known:

$$AB = 11{,}289 \text{ meters.}$$

$$\begin{array}{llll} \triangle ABC, & \triangle CBD, & \triangle DBE, & \triangle DEF, \\ \angle A = 58° 10' 35'', & \angle B = 86° 50' 0'', & \angle D = 79° 12' 8'', & \angle D = 50° 41' 5'', \\ \angle B = 69° 55' 0''; & \angle C = 46° 48' 0''; & \angle B = 73° 29' 10''; & \angle E = 45° 20' 40''; \end{array}$$

calculate EF. *Ans.* 19955 m.



89. The adjacent figure shows a chain of four triangles in which all the angles, and $AB = m$, are known. To designate the angles we use C_1, C_2, C_3 for the three angles at C, and similarly for the other vertices. Calculate in turn x_1, x_2, x_3, x_4, and show that

$$x_4 = m \frac{\sin A_1 \sin B_2 \sin C_3 \sin D_4}{\sin C_1 \sin D_2 \sin E_3 \sin F_4}.$$

(Exercise 88 gives such a chain of triangles taken from the Transcontinental Triangulation of the U. S. Geodetic Survey.)

90. A tower 50 ft. high stands on the edge of a cliff 150 ft. high. At what distance from the foot of the cliff will the tower subtend an angle of 5°? *Ans.* 59.1 or 513 ft.

91. A right triangle whose perimeter is 100 ft. rests with its hypotenuse on a plane, the vertex of the right angle being 10 ft. from the plane. The angle between the plane of the triangle and the supporting plane is 30°. Find the sides of the triangle.

92. The sides of a triangle are 100, 150, 200 ft. At the vertex of the smallest angle a line 100 ft. long is drawn perpendicular to the plane of the triangle. Find the angles subtended at the farther end of this line by the sides of the triangle.

93. An equilateral triangle 50 ft. on a side rests with one side on a plane with which its plane makes an angle of 60°. How far is the third vertex from the plane?

94. As in exercise 93, if the triangle, instead of being equilateral, has sides 40, 20, 30 ft. and rests on the shortest side. *Ans.* $\frac{45\sqrt{5}}{4}$.

95. The sides of a triangle are as 4 : 2 : 3, and the longest median is 10 ft. Find the sides and angles.

96. The following measurements of a field *ABCD* are made: *A* to *B*, due north, 10 chains; *B* to *C*, N. 30° E., 6 chains; *C* to *D*, due east, 8 chains; calculate *AD*, and the area of the field in acres. (1 chain = 4 rods.) *Ans.* 18.76 ch.; 7.578 A.

97. The following measurements of a field *ABCDE* are made: *A* to *B*, due east, 25.52 chains; *B* to *C*, E. 40° 26′ N., 22.25 chains; *C* to *D*, N. 48° 26′ W., 33.75 chains; *D* to *E*, W. 31° 15′ S., 18.32 chains; calculate *EA* and the area of the field in acres.

98. In the field of exercise 96 how much area is cut off by a line due east through *B*? *Ans.* 3.62 acres, south of dividing line.

99. In the field of exercise 97 where should an east and west line be drawn so as to bisect the area?

100. In the field of exercise 97 where should a north and south line be drawn to cut off 30 acres from the western part of the area? *Ans.* 10.892 ch. east of *A*.

101. If P be the pull required to move a weight W up a plane inclined to the horizontal at an angle i, and μ the coefficient of friction, then

$$P = W\frac{\sin i + \mu \cos i}{\cos i - \mu \sin i}.$$

Calculate P when $W = 1000$ lbs., $i = 30°$, $\mu = 0.1$.

102. In exercise 101, what is i if $P = \frac{1}{2} W$ and $\mu = 0.1$? *Ans.* $\tan^{-1}\frac{8}{19}$.

103. If l be the length of a plane inclined to the horizontal at an angle i, μ the coefficient of friction and g the acceleration due to gravity (32+ ft. per sec. per sec.) the time in seconds required by a body to slide down the plane is

$$T = \sqrt{\frac{2l}{g(\sin i - \mu \cos i)}}.$$

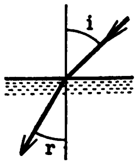
What is T when $l = 25$ ft., $i = 20°$, $\mu = 0.1$?

104. In exercise 103, find i when $l = 100$ ft., $\mu = 0.1$, $T = 5$ sec. *Ans.* 20° 7′.

105. When light passes from a rarer to a denser medium, the index of refraction μ is determined by the equation

$$\mu = \frac{\sin i}{\sin r}.$$

When $\mu = 1.2$, what must be i (angle of incidence) to give a deflection of 10°?



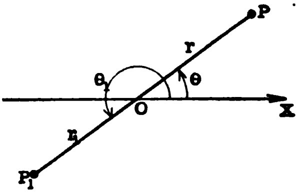
106. Find the total deflection of a ray which passes through a wedge whose angle is 30° and index of refraction 1.4, if the ray enters the wedge so that the angle of incidence is 25°, and moves in a plane ⊥ to the edge of the wedge. *Ans.* 12° 32′.

107. Solve exercise 106 when the angle of the wedge is α, the angle of incidence i, and the index of refraction μ.

CHAPTER VI

POLAR COÖRDINATES. COMPLEX NUMBERS. DE MOIVRE'S THEOREM. EXPONENTIAL VALUES OF SIN x AND COS x. HYPERBOLIC FUNCTIONS

69. Polar Coördinates. — We have made repeated use of the system of rectangular coördinates, in which the position of any point in the plane is defined by its abscissa and ordinate. A second system of coördinates defines the position of a point with reference to a single fixed line, called the **initial line,** and a fixed point on this line, called the **origin** or **pole.**



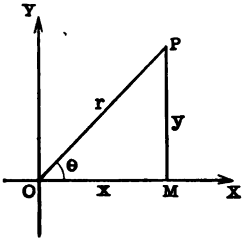
In the figure, let OX be the initial line and O the pole. We shall consider OX as the positive direction of the initial line. Let P be a point in the plane. The position of P is then fixed by its distance $OP = r$ from O, called the **radius vector,** and by the angle $XOP = \theta$, called the **vectorial angle.** Then r, θ are called the **polar coördinates** of P, and the point is indicated by (r, θ). Similarly P_1 is the point (r_1, θ_1). The coördinate θ is positive when measured counter-clockwise from OX; r is positive when measured from O along the terminal side of θ; it is negative when measured from O along the terminal side of θ produced back through O. Thus the points (5, 30°) and (−5, 210°) coincide. Similarly with (− 3, 135°) and (3, − 45°).

Exercises. Plot the following points:

$$(1, 45°);\ (-1, 45°);\ (3, 60°);\ (3, -60°);\ \left(4, \frac{\pi}{8}\right);\ \left(2, -\frac{2\pi}{5}\right);\ \left(\frac{2}{3}, \frac{5\pi}{6}\right);$$
$$\left(-\frac{2}{3}, -\frac{5\pi}{6}\right);\ \left(1, \frac{3\pi}{2}\right);\ \left(-1, -\frac{3\pi}{2}\right);\ (\pi;\ 800°).$$

Calculate the rectangular coördinates of each of these points, taking O as origin and OX as the x-axis.

70. Relation between Polar and Rectangular Coördinates. — Let O be the origin and OX the initial line of a system of polar coördinates (figure). Let OX and OY be the axes of a rectangular system of coördinates. Then



$$x = r\cos\theta, \qquad r = \sqrt{x^2 + y^2},$$
$$y = r\sin\theta; \qquad \theta = \tan^{-1}\frac{y}{x}.$$

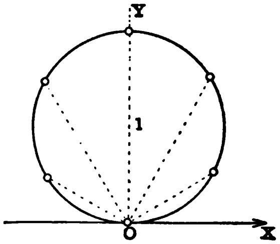
71. Curves in Polar Coördinates. — When r and θ are unrestricted, the point (r, θ) may take any position in the plane. When r and θ are connected by an equation, the point (r, θ) is in general restricted to a curve, the equation between r and θ being called the polar equation of the curve.

Example 1. Trace the curve whose polar equation is $r = \sin\theta$.

Assume a series of values for θ, calculate the corresponding values of r and plot the points whose coördinates are corresponding values of r and θ.

θ	0°	30°	60°	90°	120°	150°	180°
r	0	.5	.87	1.0	.87	.5	0

θ	210°	240°	270°	300°	330°	360°
r	−.5	−.87	−1.0	−.87	−.5	0



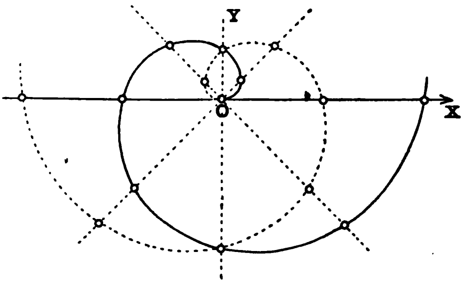
The graph is shown in the figure. For values of $\theta > 360°$, and for negative angles, no new points are obtained. The curve is a circle, with radius $= \frac{1}{2}$.

Example 2. Trace the curve $r = 2\theta$.

Here θ is understood to be in radians.

$$\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \ldots \infty.$$

$$r = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \ldots \infty.$$



For negative values of θ we get corresponding negative values of r. The curve is the double spiral in the figure, the branches shown by the full line and the dotted line being obtained from the positive and the negative values of θ respectively.

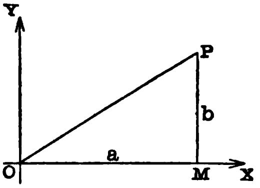
Exercises. Trace the following curves:

1. $r = 2\sin\theta$.
2. $r = \cos\theta$.
3. $r = \tan\theta$.
4. $r = \sec\theta$.
5. $r = 1 + \cos\theta$.
6. $r = 2 + \sin\theta$.
7. $r\theta = 1$.
8. $r = 2^{\theta}$.
9. $r = \cos^2\theta$.
10. $r = \cos 2\theta$.
11. $r = 4$.
12. $\theta = \frac{\pi}{4}$.

72. Complex Numbers. — Let a and b denote any two *real* numbers and $i = \sqrt{-1}$. Then the quantity $a + ib$ is called a *complex number*. It may be considered as made up of a real units and b imaginary units, $a \times 1 + b \times i$.

Real numbers can be represented by points on a straight line. To represent complex numbers geometrically, we require a plane.

Let OX and OY be a system of rectangular axes, and P a point in their plane having coördinates (a, b) (figure). Then the vector OP is considered to represent the complex number $a + ib$, and the extremity of this vector, P, is called the representative point of the complex number $a + ib$.



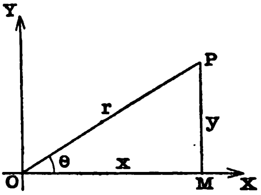
When $b = 0$, P lies on the x-axis, and the complex number reduces to a real number. Thus all points on the x-axis correspond to real numbers, and this line is called the axis of real numbers.

Let P (figure) be a point (x, y) in the plane, and let z be the complex number represented by P. Then

$$z = x + iy.$$

Now take OX as the initial line and O as the pole of a system of polar coördinates. Let the polar coördinates of P be (r, θ). Then

$$x = r\cos\theta; \qquad y = r\sin\theta.$$



Hence

$$z = x + iy = r(\cos\theta + i\sin\theta).$$

Here r is called the **modulus** and θ the **angle** of the complex number z.

When r is fixed, and θ is changed by integral multiples of 2π, we obtain a set of complex numbers of the form,

$$z = r[\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)];$$
$$n = 0, \quad \pm 1, \quad \pm 2, \ldots .$$

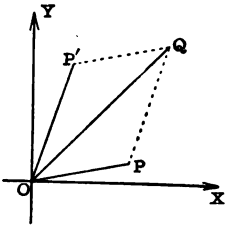
All these numbers have the same representative point.

73. Addition of Complex Numbers. — The sum of two complex numbers,

$$z = x + iy \quad \text{and} \quad z' = x' + iy',$$

we *define* by the equation

$$z + z' = (x + x') + i(y + y').$$



We proceed to consider this sum geometrically. Let P, P' (figure) be the representative points of z, z' respectively. On OP and OP' as adjacent sides construct the parallelogram $OPQP'$. Then Q is the representative point of $z + z'$. For the coördinates of Q are $(x + x', y + y')$. This amounts precisely to vector addition of the vectors OP and OP' (35).

The difference of the two complex numbers z and z' we may define by the equation

$$z - z' = (x - x') + i(y - y').$$

Exercise. Give a geometric construction for the representative point of $z - z'$.

74. Multiplication of Complex Numbers. — The product of the two complex numbers,

$$z = r(\cos\theta + i\sin\theta) \quad \text{and} \quad z' = r'(\cos\theta' + i\sin\theta'),$$

we *define* by the equation

$$zz' = rr'(\cos\theta + i\sin\theta)(\cos\theta' + i\sin\theta'),$$

the binomials to be multiplied in the usual way; thus:

$$\begin{aligned} zz' &= rr'[\cos\theta\cos\theta' - \sin\theta\sin\theta' + i(\sin\theta\cos\theta' + \cos\theta\sin\theta')] \\ &= rr'[\cos(\theta + \theta') + i\sin(\theta + \theta')]. \end{aligned}$$

Therefore the modulus of the product zz' equals the product of the moduli of z and z', and the angle of zz' equals the sum of the angles of z and z'.

By repeating this process we find

$$zz'z'' = rr'r'' [\cos(\theta + \theta' + \theta'') + i \sin(\theta + \theta' + \theta'')]$$

and so on, for any finite number of factors.

When the factors are all equal this reduces to

$$\boldsymbol{z^n = r^n (\cos n\theta + i \sin n\theta),}$$

n being a positive integer.

Exercise. Show that the above definition of the product zz' is the same as

$$zz' = xx' - yy' + i(xy' + x'y),$$

where $$z = x + iy \quad \text{and} \quad z' = x' + iy'.$$

75. De Moivre's Theorem. — When $r = 1$, then $z = \cos\theta + i\sin\theta$. Hence by the above result we have

$$\boldsymbol{(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta.}$$

This equation contains what is known as *De Moivre's Theorem.*

76. Definition of z^p. — Let p be any *real* number, positive or negative, rational or irrational. Then by analogy with the result for z^n when n is a positive integer, we *define* z^p by the equation

$$\boldsymbol{z^p = r^p (\cos p\theta + i \sin p\theta),}$$

where $$z = r(\cos\theta + i\sin\theta).$$

Then, if q also be real, we have

$$z^q = r^q(\cos q\theta + i \sin q\theta),$$

and

$$z^p z^q = r^{p+q}[\cos(p+q)\theta + i \sin(p+q)\theta] = z^{p+q}.$$

All the rules for exponents will be the same when the base is a complex number as when the base is real.

Examples.

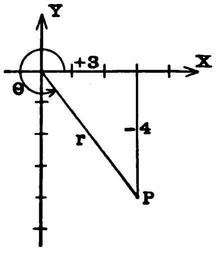
1. Find the modulus and angle of $z = 3 - 4i$.

Here $3 = r\cos\theta; \quad -4 = r\sin\theta.$

$\therefore \quad r = \sqrt{3^2 + 4^2} = 5; \ \tan\theta = \dfrac{-4}{3},$

or, $\quad \theta = \tan^{-1}\left(-\dfrac{4}{3}\right).$

The angle lies in the fourth quadrant.



2. Express $2(\cos 150° - i \sin 150°)$ in the form $x + iy$.

$$2(\cos 150° - i \sin 150°) = 2\left(-\frac{1}{2}\sqrt{3} - \frac{i}{2}\right) = -\sqrt{3} - i.$$

3. Find the value of $(1+i)^2(2-3i)$.

$$(1+i)^2 = 1 + 2i + i^2 = 2i.$$

$$(1+i)^2(2-3i) = 2i(2-3i) = 4i - 6i^2 = 6 + 4i.$$

Exercises.

1. Find the modulus and angle of

$$1-i;\quad 4+3i;\quad -5+11i;\quad 2i;\quad 2;\quad (1+i)(1-i);$$
$$3\sqrt{3}+3i;\quad (3\sqrt{3}-3i)^2;\quad (1+i\sqrt{3})(\sqrt{3}+i).$$

Give figure for each case.

2. Find the value of:

$$(1+i)^3;\quad (1-i)^4;\quad (1+i)^2(1+2i)^2;\quad (3-3i)^2(\sqrt{3}+i)^3;\quad (1-i\sqrt{3})^6.$$

77. Theorem. *If P and Q are any real quantities and if $P + iQ = 0$, then $P = 0$ and $Q = 0$.*

Proof. By hypothesis, $P + iQ = 0$ or $P = -iQ$.

Squaring, $P^2 = -Q^2$.

Now P^2 and Q^2 (if not zero) must be positive, hence the last equation states that a positive quantity equals a negative quantity. This is impossible unless both quantities are zero.

$$\therefore\quad P = 0 \quad\text{and}\quad Q = 0.$$

This theorem is used to replace a given equation of the form

$$P + iQ = 0$$

by the equivalent equations

$$P = 0;\ Q = 0.$$

As a *corollary* we have, if

$$P + iQ = P' + iQ',$$

then $P = P'$ and $Q = Q'$.

For the given equation is equivalent to $(P - P') + i(Q - Q') = 0$.

78. The *n*th Roots of Unity. — To solve the equation

$$x^n - 1 = 0,\quad\text{or}\quad x^n = 1,$$

replace 1 by its value $\cos 2k\pi + i \sin 2k\pi$, k being an integer. We obtain

$$x^n = \cos 2k\pi + i \sin 2k\pi.$$

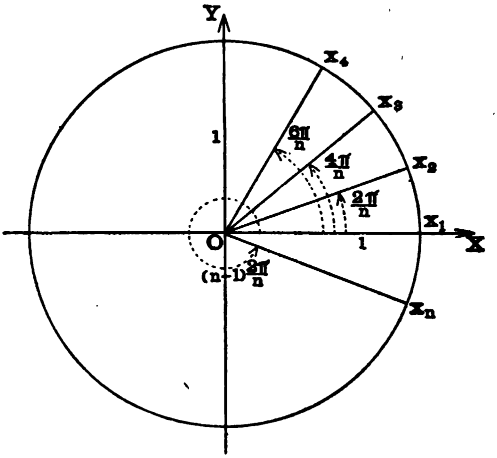
Taking the *n*th roots of both members we have, by putting $p = \frac{1}{n}$ in (76),

$$x = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}.$$

Here k may be any integer; letting $k = 0, 1, 2, \ldots n - 1$, we obtain n distinct values of x, that is, n distinct *n*th roots of 1. For other values of k we obtain the same roots over again.

Geometric Representation of the *n*th Roots of Unity. — The *n*th roots of 1 are,

$$\begin{aligned}
k &= 0; & x_1 &= \cos 0 + i \sin 0 = 1,\\
k &= 1; & x_2 &= \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n},\\
k &= 2; & x_3 &= \cos \frac{4\pi}{n} + i \sin \frac{4\pi}{n},\\
& \cdot \quad \cdot \quad \cdot & & \cdot \quad \cdot \quad \cdot \\
k &= n - 1; & x_n &= \cos \frac{2(n-1)\pi}{n} + i \sin \frac{2(n-1)\pi}{n}.
\end{aligned}$$



The representative points of $x_1, x_2, x_3, \ldots x_n$ are obtained as n equally spaced points on a circle of radius 1, the coördinates of the first point being (1, 0) (figure).

To obtain the *n*th roots of any number a, we need only multiply one of its arithmetic *n*th root by the *n*th roots of unity.

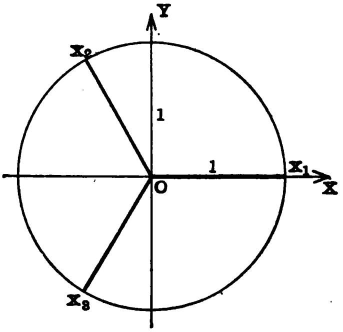
Example. Find the cube roots of unity. These are given by

$$x = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}; \quad k = 0, 1, 2.$$

$$k = 0; \quad x_1 = \cos 0^\circ + i\sin 0^\circ = 1.$$

$$k = 1; \quad x_2 = \cos 120^\circ + i\sin 120^\circ = -\frac{1}{2} + \frac{i}{2}\sqrt{3}.$$

$$k = 2; \quad x_3 = \cos 240^\circ + i\sin 240^\circ = -\frac{1}{2} - \frac{i}{2}\sqrt{3}.$$



To find the cube roots of 8, we have $\sqrt[3]{8} = 2\sqrt[3]{1} = 2;\ -1 + i\sqrt{3};\ -1 - i\sqrt{3}$. (We here use $\sqrt[3]{8}$ to denote *any* cube root of 8, not merely the principal root.)

Exercises.

1. Solve the equations $x^3 - 1 = 0$ and $x^3 - 8 = 0$ algebraically and compare with above results.

Solve the following equations by the trigonometric method and give a figure for each case:

2. $x^4 = 1$;
3. $x^4 = 81$;
4. $x^5 = 1$;
5. $x^5 = 32$;
6. $x^6 = 1$;
7. $x^6 = 27$.

79. To express sin *n*θ and cos *n*θ in terms of powers of sin θ and cos θ, *n* being a positive integer.

We have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Expand the left member by the binomial theorem, reduce all powers of i to ± 1 or $\pm i$, and group the real terms and those involving i. The above equation then becomes

$$\cos n\theta + i\sin n\theta = \left(\cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta + \cdots\right)$$
$$+ i\left(n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta + \cdots\right).$$

This equation has the form

$$P + iQ = P' + iQ'.$$

Hence by the corollary in (77) we have

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta + \cdots.$$

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta + \cdots.$$

Examples.

$$\sin 4\theta = 4\cos^3\theta \sin\theta - 4\cos\theta \sin^3\theta.$$
$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta.$$

Exercises. Expand in powers of $\sin\theta$ and $\cos\theta$:

1. $\sin 3\theta$;
2. $\cos 3\theta$;
3. $\cos 4\theta$;
4. $\sin 5\theta$;
5. $\sin 6\theta$;
6. $\cos 7\theta$.

80. Exponential Values of sin x and cos x. — We shall assume the following expansions:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots :$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.$$

These expansions are derived by the methods of Differential Calculus. (See the author's text on "Advanced Algebra and Trigonometry," under "Maclaurin's Series.") The letter e stands for an irrational number, $e = 2.7182818+$, which is the base of the *natural* system of logarithms. The last two series are used for calculating $\sin x$ and $\cos x$, by putting for x its value in *radians*. Thus, to calculate sin 10°, put $x = 10° = 0.17453$ radians. (Table IV.)

In the first series replace x by ix and *define* the result to be e^{ix}; noting that

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \ldots,$$

we obtain

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \cdots,$$
$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right).$$

Hence

$$e^{ix} = \cos x + i \sin x.$$

Replacing x by $-x$;

$$e^{-ix} = \cos x - i \sin x.$$

From these equations we find

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}; \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

These formulas are useful in many applications of the trigonometric functions.

Exercises. Using the exponential values of $\sin x$ and $\cos x$, show that:

1. $\sin^2 x + \cos^2 x = 1.$
2. $\sin 2x = 2 \sin x \cos x.$
3. $\cos 2x = \cos^2 x - \sin^2 x.$
4. $\cos^4 x - \sin^4 x = \cos^2 x - \sin^2 x.$

81. The Hyperbolic Functions. — In the expansions for $\sin x$ and $\cos x$ given at the beginning of (**80**) replace x by ix and *define* the results to be $\sin ix$ and $\cos ix$ respectively. We obtain

$$\sin ix = i\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right);$$

$$\cos ix = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.$$

These equations we consider as defining the sine and cosine of the imaginary quantity ix.

Multiply the first equation by i and subtract the result from the second. We obtain

$$\cos ix - i \sin ix = e^x.$$

Change x to $-x$;

$$\cos ix + i \sin ix = e^{-x}.$$

(Note that $\sin ix = -\sin(-ix)$ by the definition of $\sin ix$.)

Combining the last two equations by addition and subtraction, we find

$$\cos ix = \frac{e^x + e^{-x}}{2}; \quad \sin ix = i\frac{e^x - e^{-x}}{2}.$$

We now define

Hyperbolic cosine of x ($= \cosh x$) $= \cos ix$;

Hyperbolic sine of x ($= \sinh x$) $= \frac{1}{i} \sin ix$.

Then

$$\cosh x = \frac{e^x + e^{-x}}{2}; \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

These functions are related to the hyperbola somewhat as the circular functions to the circle.

Their values can be calculated readily from the values of e^x and e^{-x} given in Table VI.

The remaining hyperbolic functions are defined by the equations

$$\tanh x = \frac{\sinh x}{\cosh x}; \quad \coth x = \frac{1}{\tanh x};$$

$$\operatorname{sech} x = \frac{1}{\cosh x}; \quad \operatorname{csch} x = \frac{1}{\sinh x}.$$

Exercises. Show that:

1. $\sinh 0 = 0; \quad \cosh 0 = 1.$
2. $\sinh \pi i = 0; \quad \cosh \pi i = -1.$
3. $\sinh \frac{\pi i}{2} = i; \quad \cosh \frac{\pi i}{2} = 0.$
4. $\sinh(-x) = -\sinh x.$
5. $\cosh(-x) = \cosh x.$
6. $\cosh^2 x - \sinh^2 x = 1.$
7. $\operatorname{sech}^2 x = 1 - \tanh^2 x.$
8. $-\operatorname{csch}^2 x = 1 - \coth^2 x.$

Draw the graphs of the equations (see Table VI):

9. $y = e^x.$
10. $y = e^{-x}.$
11. $y = \cosh x.$
12. $y = \sinh x.$

CHAPTER VII

SPHERICAL TRIGONOMETRY

82. Spherical Geometry. — We devote this article to a review of some facts concerning the geometry of the sphere.

(*a*) A plane section of a sphere is a circle. When the plane passes through the center, the section is a *great circle;* otherwise a *small circle.*

(*b*) Any two great circles intersect in two diametrically opposite points and bisect each other.

(*c*) The two points on the sphere each equally distant from all the points of a circle on the sphere are called the *poles* of the circle. A great circle is 90° distant from each of its poles.

(*d*) A *spherical triangle* is a figure bounded by three circular arcs on a sphere. In this chapter we consider only triangles whose sides are arcs of great circles. Any such triangle may therefore be considered as cut from the spherical surface by the faces of a triedral angle whose vertex is at the center. The face angles of this triedral angle measure the sides of the triangle, and its diedral angles the angles of the triangle.

(*e*) If a triangle be constructed by striking arcs of great circles with the vertices of a given triangle as poles, the new triangle is called the *polar triangle* of the given one.

Let the sides of the given triangle be a, b, c; its angles A, B, C; let the sides of the polar triangle be a', b', c' and its angles A', B', C'; we assume that A is the pole of a', B of b', and C of c'; then

$$a' = 180° - A; \qquad A' = 180° - a;$$

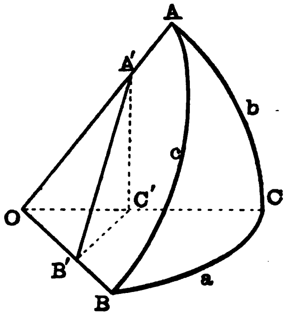
and similarly for the other sides and angles. That is, *any part of the polar triangle is the supplement of the part opposite in the given triangle.*

(*f*) The difference between the sum of the angles of a spherical triangle and 180° is called its *spherical excess.*

The area of a spherical triangle is to the area of the sphere as its spherical excess, in degrees, is to 720°. That is, if E be the spherical excess in degrees and K the area, and R the radius of the sphere, then

$$K = \frac{E}{720} \times 4\pi R^2.$$

83. Spherical Right Triangles. — Let O be the center of a sphere and ABC a triangle on its surface having $C = 90°$. The triangle shown in the figure has each part, except C, less than 90°. The results below are true in any case, as may be shown by drawing other figures, or by assuming the right triangle as a special case of the oblique triangle.



Cut the triedral angle $O - ABC$ by a plane $\perp OB$, forming the plane right $\triangle A'B'C'$, with $C' = 90°$. Then also $\triangle$s $OB'C'$ and $OB'A'$ are right-angled at B'. Further, $\angle A'B'C'$ measures $\angle B$ (**82**, (*d*)). Then

$$(a) \qquad \sin B = \sin A'B'C' = \frac{A'C'}{A'B'} = \frac{\dfrac{A'C'}{OA'}}{\dfrac{A'B'}{OA'}} = \frac{\sin b}{\sin c}.$$

$$(b) \qquad \cos B = \cos A'B'C' = \frac{B'C'}{A'B'} = \frac{\dfrac{B'C'}{OB'}}{\dfrac{A'B'}{OB'}} = \frac{\tan a}{\tan c}.$$

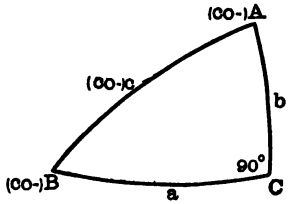
$$(c) \qquad \tan B = \tan A'B'C' = \frac{A'C'}{B'C'} = \frac{\dfrac{A'C'}{OC'}}{\dfrac{B'C'}{OC'}} = \frac{\tan b}{\sin a}.$$

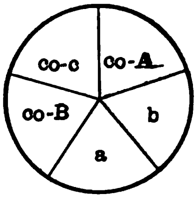
Dividing (*a*) by (*b*) and comparing with (*c*) we have

$$(d) \qquad \cos c = \cos a \cos b.$$

By combining these equations we may obtain others by which any part of the triangle may be expressed directly in terms of any two given parts, the right angle excluded. These formulas are all contained in two simple rules.

84. Napier's Rules of Circular Parts. — Let *co-x* denote the complement of any part x of the triangle. Take the complements of c, A, B, and arrange the five parts, a, b, co-A, co-c, co-B, called *circular parts* in the order in which they occur in the triangle as in the adjacent figures. Then if any one of the five be taken as the middle part, of the other four parts two will be *adjacent* and the other two *opposite* to this part. Thus, if co-c be taken as the middle part, co-B and co-A are adjacent, a and b opposite.





Rules:

$$\textit{Sine of Middle Part} = \begin{cases} \textit{Product of tangents of adjacent parts,} \\ \textit{or} \\ \textit{Product of cosines of opposite parts.} \end{cases}$$

Exercise. Taking each part in turn as the middle part write out a complete list of formulas relating to the spherical right triangle. Derive these formulas from those given above.

85. Solution of Right Triangles.

Example. Given $a = 35° 42'$; $B = 60° 25'$. Find b, c, A.

The diagram of circular parts is shown in the figure. Taking (1), (2), (3) in turn as middle part we have

(1) $\sin 35° 42' = \tan 29° 35' \tan b$;

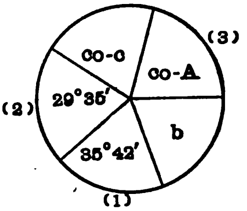
(2) $\sin 29° 35' = \tan 35° 42' \tan (\text{co-}c)$;

(3) $\sin (\text{co-}A) = \cos 29° 35' \cos 35° 42'$.

Hence,

$$\tan b = \frac{\sin 35° 42'}{\tan 29° 35'}; \quad \cot c = \frac{\sin 29° 35'}{\tan 35° 42'};$$

$$\cos A = \cos 29° 35' \cos 35° 42'.$$



Check. The computed parts must satisfy the relation

$$\sin (\text{co-}A) = \tan b \tan (\text{co-}c), \quad \text{or} \quad \cos A = \tan b \cot c.$$

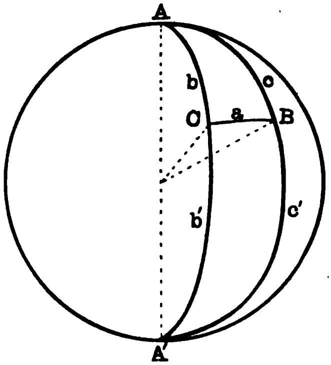
Computations.

log	log	log
$\sin 35° 42' = 9.7660$	$\sin 29° 35' = 9.6934$	$\cos 29° 35' = 9.9394$
$\tan 29° 35' = 9.7541$	$\tan 35° 42' = 9.8564$	$\cos 35° 42' = 9.9096$
$\tan b = 0.0119$	$\cot c = 9.8370$	$\cos A = 9.8490$
$b = 45° 17'$	$c = 55° 30'$	$A = 45° 4'$

Check. $\log \cos A = \log \tan b + \log \cot c.$

$$9.8490 = 0.0119 + 9.8370.$$

Ambiguous Case. When the given parts are an angle (not the right angle) and its opposite side, two solutions are possible, because the other parts are then calculated from their sines. The two triangles together form a lune, as AA' in the figure, where A, a are supposed to be the given parts.



86. Quadrantal Triangles. — A quadrantal triangle is one having a side equal to a quadrant or 90°. Its polar triangle will be a right triangle, which may be solved by Napier's Rules. The parts of the given quadrantal triangle then become known by (*e*) of (**82**).

Exercises. Solve the following triangles, C being the right angle:

1. $a = 137° 59'$,
$b = 58° 40'$.

2. $a = 137° 50'$,
$c = 64° 40'$.

3. $A = 5° 47'$,
$B = 85° 52'$.

4. $a = 41°$,
$B = 37°$.

5. $a = 134° 30'$,
$c = 122° 8'$.

6. $c = 137° 20'$,
$A = 149° 40'$.

7. $c = 73° 35'$,
$B = 101° 13'$.

8. $a = 74° 7'$,
$A = 75° 6'$.

Solve the following quadrantal triangles, side c being 90°:

9. $a = 116° 45'$,
$b = 44° 26'$.

10. $b = 36° 10'$,
$C = 65° 28'$.

11. $a = 18° 8'$,
$B = 48° 52'$.

12. $b = 35° 6'$,
$B = 33° 28'$.

13. $a = 108° 23'$,
$C = 88° 18'$.

14. $a = 80° 10'$,
$A = 68° 0'$.

87. Oblique Triangles. Two Fundamental Formulas. — We consider only triangles in which no part exceeds 180°.

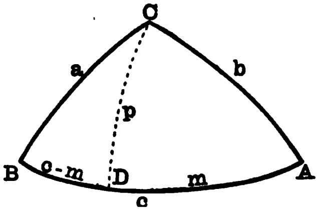
I. Law of Sines. — Let ABC be a spherical triangle. Draw $CD \perp AB$, forming two right triangles (figure).

In $\triangle ACD$, $\sin p = \sin b \sin A$.
In $\triangle BCD$, $\sin p = \sin a \sin B$.

Therefore,

$$\sin b \sin A = \sin a \sin B,$$

$$(1) \qquad \text{or} \qquad \frac{\sin a}{\sin b} = \frac{\sin A}{\sin B}.$$



That is, *the sines of the sides are proportional to the sines of the opposite angles.*

Exercise. Discuss the case where D falls on AB produced. Compare (**58**).

II. Law of Cosines. — In the figure above let $AD = m$, so that $BD = c - m$. Then in right $\triangle\ BCD$

$$\begin{aligned}\cos a &= \cos(c - m)\cos p \qquad \text{(See } (d) \text{ of } \mathbf{83})\\ &= \cos c \cos m \cos p + \sin c \sin m \cos p.\end{aligned}$$

But in $\triangle\ ACD$

$$\cos m \cos p = \cos b$$

and $$\sin m \cos p = \sin C \sin b \times \frac{\cos A}{\sin C} = \sin b \cos A.$$

Hence

(2) $$\boldsymbol{\cos a = \cos b \cos c + \sin b \sin c \cos A.}$$

That is, *the cosine of any side equals the product of the cosines of the other two sides plus the product of their sines by the cosine of their included angle.*

Exercise. Discuss the case where D falls on AB produced. Compare (**59**).

From the fundamental formulas (1) and (2) we shall derive a series of other formulas adapted to the solution of triangles.

88. Principle of Duality. — By means of (*e*) of (**82**) any formula relating to the spherical triangle can be made to yield a second formula. Thus, let $\triangle\ A'B'C'$ be polar to $\triangle\ ABC$. Then from (1) and (2), applied to $\triangle\ A'B'C'$, we have

$$\frac{\sin a'}{\sin b'} = \frac{\sin A'}{\sin B'}; \quad \cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

But $$\begin{aligned} a' &= 180° - A, & A' &= 180° - a,\\ b' &= 180° - B, & B' &= 180° - b,\\ c' &= 180° - C, & C' &= 180° - c.\end{aligned}$$

Substituting and reducing, we have

$$\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b},$$

(3) $$\boldsymbol{\cos A = -\cos B \cos C + \sin B \sin C \cos a.}$$

The first of these is simply the law of sines; the second is a new formula.

89. Formulas for the Half-Angle. — Solving (2) for cos A, we have

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

Then $\sin\frac{1}{2}A = \sqrt{\frac{1-\cos A}{2}}$. Why not $\pm\sqrt{\frac{1-\cos A}{2}}$?

$$= \sqrt{\frac{1-\dfrac{\cos a - \cos b\cos c}{\sin b \sin c}}{2}}$$

$$= \sqrt{\frac{\sin b\sin c - \cos a + \cos b\cos c}{2\sin b\sin c}}$$

$$= \sqrt{\frac{\cos(b-c) - \cos a}{2\sin b\sin c}} \qquad \textbf{(44)}$$

$$= \sqrt{\frac{2\sin\dfrac{a+b-c}{2}\sin\dfrac{a-b+c}{2}}{2\sin b\sin c}}. \qquad \textbf{(47)}$$

Now let

(4) $$2s = a + b + c;$$

then, as in (**61**),

$$\frac{a+b-c}{2} = s - c \quad\text{and}\quad \frac{a-b+c}{2} = s - b;$$

therefore,

(5) $$\sin\tfrac{1}{2}A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}.$$

Similarly, starting with $\cos\frac{1}{2}A = \sqrt{\frac{1}{2}(1+\cos A)}$, we get

(6) $$\cos\tfrac{1}{2}A = \sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}.$$

By dividing,

(7) $$\tan\tfrac{1}{2}A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}.$$

Given the three sides, one of these formulas, preferably the last, will determine the angles. When all three angles are desired, let

(8) $$\tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}};$$

then

(9) $$\tan\tfrac{1}{2}A = \frac{\tan r}{\sin(s-a)},$$

(10) $$\tan\tfrac{1}{2}B = \frac{\tan r}{\sin(s-b)},$$

(11) $$\tan\tfrac{1}{2}C = \frac{\tan r}{\sin(s-c)}.$$

90. Formulas for the Half Sides. — Proceeding as above with (3) of (88), or by applying the principle of duality to formulas (5) to (11), we have, on putting

$$2S = A + B + C \tag{12}$$

and

$$\tan R = \sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}}, \tag{13}$$

$$\sin\tfrac{1}{2}a = \sqrt{\frac{-\cos S\cos(S-A)}{\sin B\sin C}}, \tag{14}$$

$$\cos\tfrac{1}{2}a = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}}, \tag{15}$$

$$\tan\tfrac{1}{2}a = \sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}, \tag{16}$$

$$\tan\tfrac{1}{2}a = \tan R\cos(S-A), \tag{17}$$

$$\tan\tfrac{1}{2}b = \tan R\cos(S-B), \tag{18}$$

$$\tan\tfrac{1}{2}c = \tan R\cos(S-C). \tag{19}$$

91. Napier's Analogies. — Dividing $\tan\frac{1}{2}A$ by $\tan\frac{1}{2}B$ and reducing, we have

$$\frac{\tan\frac{1}{2}A}{\tan\frac{1}{2}B} = \frac{\sin(s-b)}{\sin(s-a)}.$$

By composition and division,

$$\frac{\tan\frac{1}{2}A + \tan\frac{1}{2}B}{\tan\frac{1}{2}A - \tan\frac{1}{2}B} = \frac{\sin(s-b)+\sin(s-a)}{\sin(s-b)-\sin(s-a)}.$$

Reducing tangents to sines and cosines and simplifying the resulting complex fraction, applying the formulas for $\sin(x \pm y)$ on the left and for $\sin u \pm \sin v$ on the right, we have

$$\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)} = \frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a-b)}, \tag{20}$$

or,

$$\tan\tfrac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c. \tag{20'}$$

Multiplying $\tan \frac{1}{2} A$ by $\tan \frac{1}{2} B$ and reducing,

$$\frac{\tan \frac{1}{2} A \tan \frac{1}{2} B}{1} = \frac{\sin (s - c)}{\sin c}.$$

By composition and division, and reduction as above,

$$(21) \qquad \frac{\cos \frac{1}{2} (A + B)}{\cos \frac{1}{2} (A - B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2} (a + b)},$$

or,

$$(21') \qquad \tan \frac{1}{2} (a + b) = \frac{\cos \frac{1}{2} (A - B)}{\cos \frac{1}{2} (A + B)} \tan \frac{1}{2} c.$$

These formulas determine the other two sides when two angles and their included side are given.

Proceeding as above with $\tan \frac{1}{2} a$ and $\tan \frac{1}{2} b$, or by the principle of duality applied to formulas (20) to (21′), we obtain

$$(22) \qquad \frac{\sin \frac{1}{2} (a + b)}{\sin \frac{1}{2} (a - b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2} (A - B)},$$

or,

$$(22') \qquad \tan \frac{1}{2} (A - B) = \frac{\sin \frac{1}{2} (a - b)}{\sin \frac{1}{2} (a + b)} \cot \frac{1}{2} C;$$

and,

$$(23) \qquad \frac{\cos \frac{1}{2} (a + b)}{\cos \frac{1}{2} (a - b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2} (A + B)},$$

or,

$$(23') \qquad \tan \frac{1}{2} (A + B) = \frac{\cos \frac{1}{2} (a - b)}{\cos \frac{1}{2} (a + b)} \cot \frac{1}{2} C.$$

These formulas determine the other two angles when two sides and their included angle are given.

92. Area of a Spherical Triangle. — This may be calculated by (*f*) of (**82**), namely,

$$K = \frac{E \text{ (degrees)}}{720} \times 4\pi R^2, \quad \text{or,} \quad K = E \text{ (radians)} \times R^2.$$

To obtain E, we may first calculate the angles. E may also be obtained by one of the following formulas which we add without proofs.

$$\tan \tfrac{1}{2} E = \frac{\tan \frac{1}{2} a \tan \frac{1}{2} b \sin C}{1 + \tan \frac{1}{2} a \tan \frac{1}{2} b \cos C};$$

$$\tan \tfrac{1}{4} E = \sqrt{\tan \frac{s}{2} \tan \frac{s-a}{2} \tan \frac{s-b}{2} \tan \frac{s-c}{2}}.$$

93. Solution of Spherical Oblique Triangles. — Six cases arise, according to the nature of the three given parts.

I. Given two sides and an opposite angle.

Denote the given parts by a, b, A. Calculate B by (1), then C by (22) or (23), and c by (20) or (21).

Check: $\sin b : \sin c = \sin B : \sin C,$

which involves the *computed parts.*

Ambiguous Case. Formula (1) will give two (supplementary) values for B. Two solutions are obtained when both values of B lead to values of C. Otherwise one or both values of B must be rejected.

Rule. Retain values of B which make $A - B$ and $a - b$ of like sign. Otherwise (20) and (22) take the impossible form $+ = -$.

II. Given two angles and an opposite side.

Denote the given parts by A, B, a. Calculate b by (1), then proceed as in I.

Ambiguous Case. Formula (1) gives two values of b. *Retain the value or values which make $A - B$ and $a - b$ of like sign.*

III. Given the three sides.

Calculate the angles by (9), (10), (11).

Check: $\sin A : \sin a = \sin B : \sin b = \sin C : \sin c.$

IV. Given the three angles.

Calculate the sides by (17), (18), (19).

Check: As in III.

V. Given two sides and their included angle.

Denote the given parts by a, b, C. Calculate $\frac{1}{2}(A + B)$ by (23′), $\frac{1}{2}(A - B)$ by (22′); then A and B by addition and subtraction; obtain c in two ways by the law of sines. This furnishes a check; or check by (20) or (21).

VI. Given two angles and their included side.

Denote the given parts by A, B, c. Calculate $\frac{1}{2}(a + b)$ from (21′), $\frac{1}{2}(a - b)$ from (20′); hence get a and b; obtain C in two ways by the law of sines. This gives a check; or check by (22) or (23).

94. Suggested Forms for Computations. Case I.

Example. Given $a = 100° 37'$, $b = 62° 25'$, $A = 120° 48'$.

Formulas.

$$\sin B = \frac{\sin b}{\sin a} \sin A,$$

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(A-B),$$

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a-b).$$

Check:

$$\frac{\sin b}{\sin c} = \frac{\sin B}{\sin C}.$$

Computations.

$$\begin{array}{rl rl rl}
\log \sin b = & 9.9476 & a = & 100° \, 37' & A = & 120° \, 48' \\
\log \sin A = & 9.9340 & b = & \underline{62° \, 25'} & B = & \underline{50° \, 46'} \\
\text{colog} \sin a = & \underline{0.0075} & a + b = & 162° \, 62' & A + B = & 170° \, 94' \\
\log \sin B = & 9.8891 & a - b = & 38° \, 12' & A - B = & 70° \;\; 2' \\
B \doteq & 50° \, 46'.5 & \frac{1}{2}(a+b) = & 81° \, 31' & \frac{1}{2}(A+B) = & 85° \, 47' \\
\text{or} & 129° \, 13'.5 & \frac{1}{2}(a-b) = & 19° \;\; 6' & \frac{1}{2}(A-B) = & 35° \;\; 1'
\end{array}$$

Reject the larger value of B by the rule in I.

$$\begin{array}{rl rl}
\log \tan \frac{1}{2}(A - B) = & 9.8455 & \log \tan \frac{1}{2}(a - b) = & 9.5395 \\
\log \sin \frac{1}{2}(a + b) = & 9.9952 & \log \sin \frac{1}{2}(A + B) = & 9.9989 \\
\text{colog} \sin \frac{1}{2}(a - b) = & \underline{0.4852} & \text{colog} \sin \frac{1}{2}(A - B) = & \underline{0.2412} \\
\log \cot \frac{1}{2} C = & 0.3259 & \log \tan \frac{1}{2} c = & 9.7796 \\
\frac{1}{2} C = & 64° \, 43'.5 & \frac{1}{2} c = & 31° \, 3' \\
C = & 129° \, 27' & c = & 62° \, 6' \\
\textit{Check:} \; \log \sin b = & 9.9476 & \log \sin B = & 9.8891 \\
\sin c = & \underline{9.9463} & \sin C = & \underline{9.8877} \\
 & 0.0013 & & 0.0014
\end{array}$$

Note. In the solutions of triangles, a complete form should be prepared in advance, so that only numerical values need be inserted when the tables are opened.

Case II. Given A, B, a. To find b, c, C.

Formulas.

$$\sin b = \frac{\sin B}{\sin A} \sin a.$$

The rest of the calculations are as in Case I.

Case III. Given a, b, c. To find A, B, C.

Formulas.

$$\tan \tfrac{1}{2} A = \frac{\tan r}{\sin (s - a)}; \quad \tan \tfrac{1}{2} B = \frac{\tan r}{\sin (s - b)}; \quad \tan \tfrac{1}{2} C = \frac{\tan r}{\sin (s - c)};$$

$$s = \frac{a+b+c}{2}; \quad \tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.$$

Form for Computations. The numbers indicate the order of the entries after the form has been prepared.

$a =$	(1)	$\log\sin(s-a) =$	(9)	$\log\tan\frac{1}{2}A =$	(15)
$b =$	(2)	$\log\sin(s-b) =$	(10)	$\log\tan\frac{1}{2}B =$	(16)
$c =$	(3)	$\log\sin(s-c) =$	(11)	$\log\tan\frac{1}{2}C =$	(17)
$2s =$	(4)	$\operatorname{colog}\sin s =$	(12)	$\frac{1}{2}A =$	(18)
$s =$	(5)	$2\,\rfloor$	(13)	$\frac{1}{2}B =$	(19)
$s-a =$	(6)	$\log\tan r =$	(14)	$\frac{1}{2}C =$	(20)
$s-b =$	(7)				
$s-c =$	(8)				
$\therefore\ A =$	(21);	$B =$	(22);	$C =$	(23).

Check.

$\log\sin A =$	(24)	$\log\sin B =$	(26)	$\log\sin C =$	(28)
$\log\sin a =$	(25)	$\log\sin b =$	(27)	$\log\sin c =$	(29)
Difference $=$	(30)		(31)		(32)

The last three numbers should agree.

Case IV. Given A, B, C. To find a, b, c.

Modify the form for Case III so as to apply to formulas (17), (18), (19) of (90).

Case V. Given a, b, C. To find A, B, c.

Formulas.

$$\tan\tfrac{1}{2}(A+B) = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C;$$

$$\tan\tfrac{1}{2}(A-B) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C;$$

$$\sin c = \frac{\sin C}{\sin A}\sin a = \frac{\sin C}{\sin B}\sin b.$$

Check. The two values of c should agree. This does not guard against an error in $\log\cot\frac{1}{2}C$ or in $\log\sin C$, which should be carefully checked. A complete check is furnished by (20).

Form for Computations.

$a =$	(1)	$\log\cos\frac{1}{2}(a-b) =$	(5)	$\log\sin\frac{1}{2}(a-b) =$	(6)
$b =$	(2)	$\operatorname{colog}\cos\frac{1}{2}(a+b) =$	(7)	$\operatorname{colog}\sin\frac{1}{2}(a+b) =$	(8)
$a+b =$	(3)	$\log\cot\frac{1}{2}C =$	(9)	$\log\cot\frac{1}{2}C =$	(9)
$a-b =$	(4)	$\log\tan\frac{1}{2}(A+B) =$	(10)	$\log\tan\frac{1}{2}(A-B) =$	(11)
$\frac{1}{2}(A+B) =$	(12)	$\log\sin C =$	(16)	$\log\sin C =$	(16)
$\frac{1}{2}(A-B) =$	(13)	$\log\sin a =$	(17)	$\log\sin b =$	(19)
$A =$	(14)	$\operatorname{colog}\sin A =$	(18)	$\operatorname{colog}\sin B =$	(20)
$B =$	(15)	$\log\sin c =$	(21)	$\log\sin c =$	(22)
$c =$	(23)				

Check. (21) should equal (22). Again look up $\log\cot\frac{1}{2}C$ and $\log\sin C$.

Case VI. Given A, B, c. To find a, b, C.

Arrange form as in Case V, to apply to formulas (20′) and (21′).

If desired, Cases II, IV, VI may be dropped entirely because they reduce to Cases I, III, V, respectively by use of the polar triangle.

95. Alternative Method under Cases V and VI.

Case V. When two sides and their included angle are given, and the third side only is required, the best procedure is as follows:

Given b, c, A. *To find* a.

$$(1)\qquad \cos a = \cos b \cos c + \sin b \sin c \cos A.$$

To adapt this to calculation by logarithms, let

$$(2)\qquad \cos c = n \cos N, \quad \text{and} \quad \sin c \cos A = n \sin N;$$

these give

$$(3)\qquad \tan N = \tan c \cos A, \quad \text{and} \quad n = \sec N \cos c.$$

Equations (2) substituted in (1) give

$$(4)\qquad \cos a = n \cos (b - N) = \cos c \sec N \cos (b - N).$$

To compute a, get N from (3), then a from (4).

Case VI. When two angles and their included side are given, and the third angle only is required, proceed as follows:

Given B, C, a. *To find* A.

$$(1')\qquad \cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

To adapt this to logarithmic calculation let

$$(2')\qquad \cos C = n \cos N, \quad \text{and} \quad \sin C \cos a = n \sin N.$$

These give

$$(3')\qquad \tan N = \tan C \cos a; \quad n = \sec N \cos C.$$

Reducing (1′) by means of (2′) and (3′), we have

$$(4')\qquad \cos A = -n \cos (b + N) = -\cos C \sec N \cos (b + N).$$

Calculate N from (3′), then A from (4′).

96. Exercises. Solve the triangles whose given parts are:

1.	**2.**	**3.**	**4.**
$a = 70° 5'$,	$a = 82° 40'$,	$a = 150° 20'$	$a = 115° 13'.4$,
$b = 63° 22'$,	$b = 84° 20'$,	$b = 137° 20'$,	$b = 127° 17'.8$,
$c = 59° 17'$.	$c = 114° 30'$.	$c = 20° 6'$.	$c = 57° 48'.9$.

5.
$a = 54^\circ\,40'$,
$c = 131^\circ\,30'$,
$B = 96^\circ\,47'$.

6.
$a = 51^\circ\,15'$,
$b = 149^\circ\,25'$,
$B = 139^\circ\,51'$.

7.
$b = 112^\circ\,0'.3$,
$c = 95^\circ\,13'.3$,
$A = 83^\circ\,35'.5$.

8.
$a = 63^\circ\,51'.5$,
$b = 144^\circ\,13'.4$,
$C = 128^\circ\,58'.8$.

9.
$a = 132^\circ\,39'$,
$B = 52^\circ\,38'$,
$C = 41^\circ\,40'$.

10.
$c = 51^\circ\,44'$,
$A = 91^\circ\,36'$,
$B = 123^\circ\,12'$.

11.
$a = 123^\circ\,43'.8$,
$B = 127^\circ\,41'.8$,
$C = 83^\circ\,39'.3$.

12.
$b = 47^\circ\,42'$,
$A = 91^\circ\,47'.7$,
$C = 55^\circ\,52'.7$.

13
$A = 55^\circ\,7'$,
$B = 148^\circ\,41'$,
$C = 24^\circ\,25'$.

14.
$A = 72^\circ\,52'$,
$B = 123^\circ\,40'$,
$C = 101^\circ\,45'$.

15.
$A = 108^\circ\,45'$,
$B = 140^\circ\,50'$,
$C = 139^\circ\,25'$.

16.
$A = 80^\circ\,19'.2$,
$B = 115^\circ\,36'.8$,
$C = 79^\circ\,10'.5$.

17.
$b = 90^\circ\,36'$,
$c = 39^\circ\,40'$,
$B = 50^\circ\,52'$.

18.
$a = 114^\circ\,27'$,
$b = 84^\circ\,22'$,
$B = 80^\circ\,19'$.

19.
$a = 118^\circ\,22'$,
$b = 40^\circ\,5'.6$,
$B = 29^\circ\,42'.6$.

20.
$a = 143^\circ\,39'.7$,
$b = 133^\circ\,29'.3$,
$A = 137^\circ\,44'.7$.

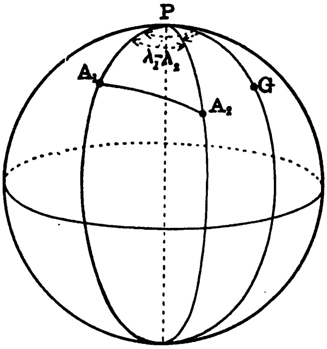
21.
$a = 137^\circ\,30'$,
$A = 125^\circ\,0'$,
$C = 41^\circ\,50'$.

22.
$a = 35^\circ\,37'.3$,
$A = 29^\circ\,3'$,
$B = 45^\circ\,44'.1$.

23.
$a = 135^\circ\,37'.8$,
$A = 129^\circ\,14'.7$,
$B = 110^\circ\,47'.3$.

24.
$a = 126^\circ\,17'.3$,
$A = 117^\circ\,44'.6$,
$C = 26^\circ\,50'.4$.

97. Applications to the Terrestrial Sphere. — We shall consider the earth as a sphere with a radius of 3960 miles. Longitudes are to be reckoned from Greenwich westward through 360° or 24^h. We shall denote longitude by λ, latitude by ϕ.



Problem 1. Given the latitudes and longitudes of two stations, to find the distance between them.

Let P be the earth's north pole, G Greenwich, A_1 and A_2 the two stations (figure). Let the positions of the two stations be λ_1, ϕ_1 and λ_2, ϕ_2 respectively.

Then in $\triangle A_1PA_2$, $PA_1 = 90^\circ - \phi_1$, $PA_2 = 90^\circ - \phi_2$, and $\angle A_1PA_2 = \lambda_1 - \lambda_2$. Hence in $\triangle A_1PA_2$ two sides and their included angle are known, and A_1A_2 (in degrees) may be calculated as in Case V of (**95**).

Problem 2. A ship is to sail from A_1 to A_2 by the shortest path (great circle). On what course (at what angle with the meridian) will she depart from A_1; on what course will she arrive at A_2?

Assuming the positions of A_1 and A_2 given, we have two sides and the included angle of the triangle A_1PA_2. We must calculate angles A_1 and A_2. This comes under Case V of (**94**).

Exercises.

1. Calculate the sides (in miles), the angles, and the area (in square miles) of the triangle whose vertices are:

	h m s	
New York	$\lambda = 4\ 55\ 54$,	$\phi = 40° 45'$ N.
San Francisco	$8\ 9\ 43$,	$37° 47'$ N.
Mexico City	$6\ 36\ 27$,	$19° 26'$ N.

2. Calculate the sides (in miles), the angles, and the area (in square miles) of the triangle whose vertices are:

	h m s	
New York	$\lambda = 4\ 55\ 54$,	$\phi = 40° 45'$ N.
Rio de Janeiro	$2\ 52\ 41$,	$22° 54'$ S.
Liverpool	$0\ 12\ 17$,	$53° 24'$ N.

3. Find the distance along the great circle from Boston, $\lambda = +4^h\ 44^m\ 15^s$, $\phi = 42° 22'$ N., to Wellington in New Zealand, $\lambda = -11^h\ 39^m\ 7^s$, $\phi = 41° 8'$ S.

4. A vessel sails on a great circle from San Francisco, $\lambda = 8^h\ 9^m\ 43^s$, $\phi = 37° 47'$ N. to Sydney, $\lambda = 13^h 55^m 10^s$, $\phi = 33° 52'$ S. Find the courses of departure and arrival and the distance sailed.

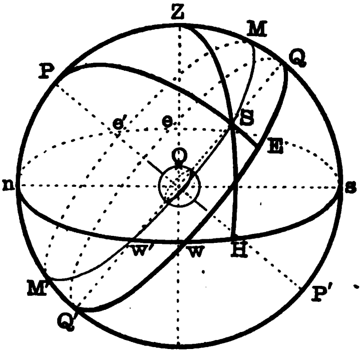
5. If the vessel in exercise 2 makes 12 knots an hour, what is her position (λ and ϕ) and on what course is she sailing 5 days after leaving San Francisco? (1 knot = 1 nautical mile = $1'$ on a great circle.)

6. Find the shortest distance between two points on the Arctic circle which differ by one hour in longitude. How far is it between these points on the Arctic circle?

7. If a person were to start from a point in 80° north latitude and go always directly east for a distance of 100 miles, how much would he change his longitude? What would be the shortest distance between the beginning and end points of his journey? (Note that to go always directly east is to go along a parallel of latitude.)

98. Applications to the Celestial Sphere. — For the purpose of this article we assume the *celestial sphere* to be an indefinitely large sphere concentric with that of the earth. On it as a background we see all celestial objects.

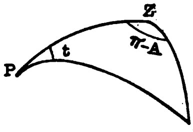
The projections on the celestial sphere of the earth's poles, equator, meridians and parallels of latitude are named respectively the *celestial poles* (P, P' in the figure), the *celestial equator* or simply *equator* ($QwQ'e$), *hour circles* (as PSE), and *parallels of declination* (as MSM').



An observer at O on the earth's surface will have his *zenith* at Z, where the plumb line at O, if produced, would meet the celestial sphere; his *horizon* is the great circle *swne*, whose pole is Z; his *meridian* is the great circle $nPZQs$, meeting the horizon in the north and south points.

Let S be a point on the celestial sphere, as the sun's center, or a star. Because of the rotation of the earth, S will appear to describe the parallel $e'MSw'M'e'$, rising at e' and setting at w'. When S has the position shown in the figure, HS is its *altitude*, denoted by h (height above horizon); $\angle\ sZH$ (measured by arc sH) is its *azimuth*, denoted by A; ZS, or $90° - h$, is the *zenith distance* of S and denoted by z. Thus h and A, or z and A, completely define the position of S with reference to horizon and zenith.

With reference to the equator and pole, ES is called the *declination* of S, denoted by δ, and $\angle\ QPE$ (angle which hour circle PS of S makes with meridian PQ) is called its *hour angle*, denoted by t; PS or $90° - \delta$ is the *polar distance* of S, and denoted by p. Thus the position of S is defined by δ and t, or by p and t.



$\triangle\ PZS$ is called the *astronomical triangle;* its parts, except the angle at S which we shall not need, are:

$$
\begin{aligned}
PZ &= 90° - nP = 90° - \phi;\\
(\phi &= \text{latitude of } O.)\\
PS &= p = 90° - \delta;\\
ZS &= z = 90° - h;\\
\angle\ ZPS &= t;\\
\angle\ PZS &= 180° - A.
\end{aligned}
$$

Problem 1. *Given the latitude of O, and the declination and altitude of S, calculate the hour angle and azimuth of S.*

Here the three sides of △ PZS are known, and it is only necessary to calculate the angles at P and Z (**III, 93**).

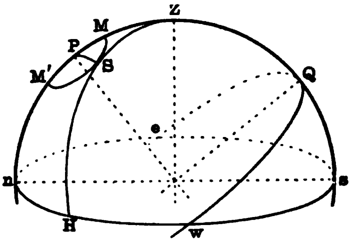
Problem 2. *In a given latitude, and for a given declination of the sun, find the sun's hour angle at sunset and the length of day (sunrise to sunset).*

Here S is on the horizon and PZS a quadrantal triangle. We obtain t by solving the polar right triangle for $180 - t$. The length of day will be $2\,t$.

Problem 3. *Given the sun's declination and its hour angle when it bears due west* ($A = 90°$), *find the latitude.*

Here PZS is a right triangle, with the right angle at Z; p and t are known, and PZ may be calculated by use of Napier's Rules.

Problem 4. *Find the hour angle and azimuth of Polaris when at greatest elongation, given the declination of the star and the latitude of the station of observation.*



Let MSM' be the star's diurnal path about the pole (figure). When the star is at *greatest elongation*, the great circle ZS is tangent to the small circle MSM', of which PS is a radius. Hence △ PZS is right-angled at S; PZ and PS are known, and the angles at P and Z may be found by aid of Napier's Rules.

Exercises.

1. In latitude 40° 49′ the sun's altitude is observed to be 20° 20′; its declination is 15° 12′; find its azimuth and hour angle.

2. With latitude and declination as in exercise 1, find the sun's hour angle when it is due west; when it sets; find its azimuth at sunset; find the length of day.

3. With latitude and declination as in exercise 1, find the sun's altitude and azimuth when its hour angle is 45°.

4. The sun, in declination 12° 22′, is observed to have an altitude of 30° when due west. What is the latitude of the station?

5. The declination of Polaris being 88° 49′, find his azimuth and hour angle at greatest elongation at a station in latitude 40° 49′.

6. As in exercise 5 for the star 51 Cephei, $\delta = 87° 11'$, and for δ Ursæ Minoris, $\delta = 86° 37'$.

7. The stylus of a horizontal sundial consists of a rod pointing to the north celestial pole. Hence its shadow falls due north when the sun is on the meridian, that is, at apparent noon. What angle does its shadow make with the meridian one hour after apparent noon, at a place in latitude 40°?

(*Suggestion.* In the first figure of this article let $nP = 40°$ and $\angle ZPS = 1^h$ or 15°. The stylus lies in the line $P'P$, and its shadow, cast by the sun S, must lie in the plane $SP'P$, and hence will fall on the plane of the dial, *swne*, along the line of intersection of these two planes. This line will be determined by the center of the sphere and the point where arc SP produced will meet arc *ne*. Call this point S'. Then arc nS' measures the required angle, and may be found by solving right $\triangle nPS'$, in which $nP = 40°$ and $\angle nPS' = 15°$).

8. What angle does the shadow of a horizontal sundial make with its noon position t hours after noon in latitude ϕ? (*Ans.* $\tan x = \tan t \sin \phi$, x being the required angle.)

9. Calculate the angles which the hour lines of a horizontal sundial make with the noon-line in an assumed latitude.

ANSWERS

(Answers to odd-numbered exercises only)

Article 8 — Page 6

1. *b, c, B*: 142.8, 174.3, 55°. **3.** *a, b, B*: 55.8, 50.2, 42°. **5.** *a, c, B*: 470, 886, 32°. **7.** *a, b, B*: .0345, .0289, 40°. **9.** *b, c, A*: 21.4, 27.2, 38°. **11.** 81.9 ft. **13.** 104.58 ft. **15.** 291.2 ft. **17.** 35°.

Article 19 — Page 19

	sine	cosine	tangent	cotangent	secant	cosecant
1.	cos 50°	−sin 50°	−cot 50°	−tan 50°	−csc 50°	sec 50°
3.	−sin 55°	−cos 55°	tan 55°	cot 55°	−sec 55°	−csc 55°
5.	cos 85°	−sin 85°	−cot 85°	−tan 85°	−csc 85°	sec 85°
7.	−sin 65°	−cos 65°	tan 65°	cot 65°	−sec 65°	−csc 65°
9.	sin 42°	cos 42°	tan 42°	cot 42°	sec 42°	csc 42°
11.	sin 50°	cos 50°	tan 50°	cot 50°	sec 50°	csc 50°
13.	sin 40°	cos 40°	tan 40°	cot 40°	sec 40°	csc 40°
15.	−cos 85°	sin 85°	−cot 85°	−tan 85°	csc 85°	−sec 85°
17.	sin 15°	cos 15°	tan 15°	cot 15°	sec 15°	csc 15°
19.	cos 20°	−sin 20°	−cot 20°	−tan 20°	−csc 20°	sec 20°
21.	sin 25°	cos 25°	tan 25°	cot 25°	sec 25°	csc 25°
23.	cos 20°	sin 20°	cot 20°	tan 20°	csc 20°	sec 20°
25.	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	-2	$\frac{2}{\sqrt{3}}$
27.	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	2
29.	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$\frac{2}{\sqrt{3}}$	-2
31.	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	2	$-\frac{2}{\sqrt{3}}$
33.	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	-2	$\frac{2}{\sqrt{3}}$
35.	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	-2	$-\frac{2}{\sqrt{3}}$

Article 23 — Page 21

1. 15°, 300°, 56½°, 288°, 264°. **3.** 60°, 85° 56′ 37″, 81° 17′ 45″, 21° 48′ 10″, 80° 48′ 35″. **5.** 16° 33′ 36″, 264° 3′, 110° 48′ 13″. **7.** 25 π/12, − π/8, 25 π/24, 1.85005, 1.63625. **9.** .002909, .000048, .21091, .37703. **11.** 3. **13.** 100 π/3 or 104.7+. **17.** 30′ 56″; 31′ 58″.

Article 27 — Page 26

1. $n\pi + \pi/4$; $2n\pi - \pi/6$, $(2n+1)\pi + \pi/6$; $2n\pi$, $2n\pi \pm \pi/3$. **3.** $228° 35' \pm 2n\pi$, $311° 25' \pm 2n\pi$; $2n\pi \pm 114° 28'.7$; $n\pi + 71° 34'$; $19° 28' \pm 2n\pi$, $160° 32' \pm 2n\pi$. **5.** $n\pi + 57° 15'$; $2n\pi - 15° 47'.4$, $(2n+1)\pi + 15° 47'.4$; $(2n+1)\pi \pm 78° 28'$.

Article 29 — Page 28

	sin	cos	tan	csc	sec	cot
1.	$-\frac{2}{3}$	$\pm\frac{\sqrt{5}}{3}$	$\pm\frac{2\sqrt{5}}{5}$	$-\frac{3}{2}$	$\pm\frac{3\sqrt{5}}{5}$	$\pm\frac{\sqrt{5}}{2}$
3.	$\pm\frac{3}{5}$	$\pm\frac{4}{5}$	$-\frac{3}{4}$	$\pm\frac{5}{3}$	$\pm\frac{5}{4}$	$-\frac{4}{3}$
5.	$\pm\frac{1}{2}$	$\pm\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	± 2	$\pm\frac{2\sqrt{3}}{3}$	$-\sqrt{3}$
7.	$\pm\frac{40}{41}$	$-\frac{9}{41}$	$\pm\frac{40}{9}$	$\pm\frac{41}{40}$	$-\frac{41}{9}$	$\pm\frac{9}{40}$
9.	$-\frac{4}{5}$	$\pm\frac{3}{5}$	$\pm\frac{4}{3}$	$-\frac{5}{4}$	$\pm\frac{5}{3}$	$\pm\frac{3}{4}$
11.	$-\frac{1}{m}$	$\pm\frac{\sqrt{m^2-1}}{m}$	$\pm\frac{1}{\sqrt{m^2-1}}$	$-m$	$\pm\frac{m}{\sqrt{m^2-1}}$	$\pm\sqrt{m^2-1}$
13.	$1+h$	$\pm\sqrt{-2h-h^2}$	$\pm\frac{1+h}{\sqrt{-2h-h^2}}$	$\frac{1}{1+h}$	$\pm\frac{1}{\sqrt{-2h-h^2}}$	$\pm\frac{\sqrt{-2h-h^2}}{1+h}$
15.	$\pm\frac{a^2-b^2}{2ab}$	$\frac{a^2+b^2}{2ab}$	$\pm i\frac{a^2-b^2}{a^2+b^2}$	$\pm\frac{2abi}{a^2-b^2}$	$\frac{2ab}{a^2+b^2}$	$\pm i\frac{a^2+b^2}{a^2-b^2}$

Article 30 — Page 30

3. $\frac{1 \pm \sqrt{1-\sin^2 x}}{\sin^2 x}$. **5.** $\frac{2\csc^2\theta}{\csc^2\theta - 1}$.

Article 33 — Page 33

1. 650 ft. **3.** 1418 ft.

Article 36 — Page 36

1. (68.4 lb., 39° 19′). **3.** (42.5 lb., 124° 15′).

Article 38 — Page 37

7. 34.53 ft. **9.** 24 in. from southern edge; angle 36° 52′. **11.** W. 30° 22′ S., 111.5 mi.

Article 48 — Page 50

1. 1; 0. **3.** $\frac{156}{205}$; $\frac{133}{205}$. **5.** $\pm\frac{\sqrt{5} \pm 4\sqrt{2}}{9}$; $\pm\frac{\sqrt{5} \mp 4\sqrt{2}}{9}$. **7.** $\pm\frac{\sqrt{7}}{111}(65 + 32\sqrt{3})$, $\pm\frac{\sqrt{7}}{111}(65 - 32\sqrt{3})$. **9.** $\frac{10487}{13325}$. **11.** $\frac{16}{65}$. **13.** $\frac{2016}{4225}$, $\frac{253}{204}$. **15.** $\frac{a^2}{2}$. **17.** $\frac{1}{2}$. **19.** $\sin 202\tfrac{1}{2}° = \frac{1}{2}\sqrt{2 - \sqrt{2}}$; $\cos 202\tfrac{1}{2}° = -\frac{1}{2}\sqrt{2 + \sqrt{2}}$; $\tan 202\tfrac{1}{2}° = \sqrt{3 - 2\sqrt{2}}$.

Article 49 — Page 56

1. .0814; .0582. **3.** 8−; 5+.

Article 55 — Page 63

1. $2n\pi \pm 60°$. **3.** $2n\pi + 3\pi/4$. **5.** $n\pi + 45°$; $n\pi + 71° 34'$. **7.** $2n\pi + 36° 52'$; $2n\pi + 90°$. **9.** $n\pi$. **11.** $n\pi$; $n\pi \pm \frac{\pi}{4}$. **13.** $n\pi$; $n\pi \pm \frac{\pi}{6}$. **15.** $(2n+1)\frac{\pi}{8}$. **17.** $\frac{2n\pi + 90°}{q \pm p}$. **19.** $\frac{2n\pi}{r-s}$; $\frac{(2n+1)\pi}{r+s}$. **21.** $n\pi$; $135° + n\pi$. **23.** $2n\pi - 60°$; $2n\pi - 120°$. **25.** $2n\pi - 30°$; $(2n+1)\pi + 30°$. **27.** $\frac{n\pi}{2}$; $n\pi + \frac{\pi}{4}$. **29.** $\frac{n\pi}{3}$; $(2n+1)\frac{\pi}{2} \pm 30°$.

Article 57 — Page 65

1. $r = \pm 5$, $\theta = \tan^{-1} \frac{4}{3}$. **3.** $r = \pm 41$, $\theta = \tan^{-1} \frac{40}{9}$. **5.** $r = \pm \frac{\sqrt{2}}{5}$, $\theta = \tan^{-1} 1$. **7.** $r = \pm \frac{1}{5}\sqrt{5}$, $\theta = \tan^{-1}(-3)$. **9.** $r = 5\sqrt{2}$, $\phi = \tan^{-1} \frac{4}{3}$, $\theta = \tan^{-1} 1$. **11.** $x^2 + y^2 = r^2$. **13.** $\frac{x^{\frac{1}{2}}}{a^2} + \frac{y^{\frac{1}{2}}}{b^2} = 1$. **15.** $x^2 + y^2 + z^2 = 1$.

Article 63 — Page 73

1. b, c, C: 1260.6, 1069.3, 55°. **3.** a, c, A: 5.000, 7.351, 38° 0′. **5.** a, b, A: 6756, 5802, 87° 40′.

Article 64 — Page 74

1. c, A, B: 851, 54° 28′, 65° 32′. **3.** c, A, B: 497, 26°, 52° 20′. **5.** a, B, C: .03826, 44° 27′, 99° 25′.

Article 65 — Page 77

1. c, B, C: 7.29, 33° 28′, 119° 14′; or c', B', C': 59.16, 146° 32′, 6° 10′. **3.** b, B, C: 72.42, 32° 8′, 89° 45′; or b', B', C': 69.98, 31° 38′, 90° 15′. **5.** a, A, B: 2232, 35° 15′, 21° 6′.

Article 66 — Page 78

1. A, B, C: 74° 40′, 47° 46′.4, 57° 33′.4. **3.** A, B, C: 94° 31′.7, 32° 2′.9, 43° 25′.4. **5.** A, B, C: 24° 42′, 30° 22′, 124° 56′.

Article 68 — Page 79

1. a, c, C: 3674, 5780, 70° 58′. **3.** a, b, B: 1566, 1068, 42° 27′. **5.** c, A, B: .1776, 76° 20′, 44° 53′. **7.** c, B, C: 156.1, 26° 43′, 131° 56′; or c', B', C': 19.57, 153° 17′, 5° 22′. **9.** A, B, C: 149° 49′, 3° 2′, 27° 9′. **11.** b, B, C: 3.588, 146° 43′, 14° 4′. **13.** b, A, B: .2729, 39° 37′, 117° 51′; or, b', A', B':

0.907, 140° 23′, 17° 5′. **15.** a, B, C: .00251, 70° 17′, 51° 50′. **17.** b, B, C: .000662, 83° 33′, 32° 36′. **19.** a, A, B: 1.235, 162° 22′, 7° 5′. **21.** a, c, C: 57260, 117570, 151° 19′.6. **23.** b, A, C: .01068, 81° 52′, 55° 41′. **25.** a, c, A: .1426, .2510, 34° 32′.1. **27.** c, B, C: 584.1, 51° 9′, 87° 38′; or c', B', C': 100.9, 128° 51′, 9° 56′. **29.** c, B, C: 1203, 31° 58′, 120° 44′; or, c', B', C': 113.8, 148° 2′, 4° 40′. **31.** a, B, C: 2496, 100° 10′, 27° 38′.8. **33.** c, A, C: 18463, 39° 39′.6, 90°. **35.** a, b, B: .03108, .01073, 14° 15′.5. **37.** a, c, C: 43.74, 64.58, 32° 19′.7. **39.** c, A, C: .005708, 79° 20′, 37° 0′; or, c', A', C': .002561, 100° 40′, 15° 40′. **51.** 7, $\sqrt{129}$, $20\sqrt{3}$. **53.** $\frac{7}{8}$. **55.** 45°, 60°, 75°; 612.5, 683 ft. **57.** 261.5. **71.** 1.239 mi. **73.** 1066 ft. **75.** $40\sqrt{5}$ ft. **77.** 45° 3′. **79.** 698.3 ft. **81.** 22.3, 70.6 ft. **85.** 62 ft. **87.** 1142 ft. **91.** 25, $33\frac{1}{3}$, $41\frac{2}{3}$ ft. **93.** 37.5 ft. **95.** 28° 57′, 46° 34′, 104° 29′; 5.892, 8.838, 11.784. (The exact values of the sides are $20\sqrt{2/23}$, $30\sqrt{2/23}$, $40\sqrt{2/23}$.) **97.** 27.35 ch.; 97.46 A. **99.** 14.4 ch. north of AB. **101.** 718.7 lb. **103.** 2.51 sec. **105.** 48° 53′. **107.** *Total defl.* $= (i - r) + (i' - r')$, where $r = \sin^{-1}\left(\frac{\sin i}{\mu}\right)$, $r' = \alpha - r$, and $i' = \sin^{-1}(\mu \sin r')$.

Article 76 — Page 91

1. $\sqrt{2}$, $-45°$; 5, 36° 52′; $\sqrt{146}$, 114° 27′; 2, 90°; 2, 0°; 2, 0°; 6, 30°; 36, $-60°$; 4, 90°.

Article 78 — Page 93

3. ± 3; $\pm 3i$. **5.** $x_1 = 2$; $x_2 = 2(\cos 72° + i \sin 72°)$; $x_3 = 2(\cos 144° + i \sin 144°)$; etc. **7.** $x_1 = \sqrt{3}$; $x_2 = \frac{\sqrt{3} + 3i}{2}$; $x_3 = \frac{-\sqrt{3} + 3i}{2}$; etc.

Article 86 — Page 100

1. c, a, B: 112° 43′.6, 133° 28′.4, 67° 49′.7. **3.** a, b, c: 4° 2′.8, 44° 17′.1, 44° 25′.9. **5.** b, A, B: 40° 40′, 122° 37′.9, 50° 18′.4. **7.** a, b, A: 146° 32′.5, 109° 48′, 144° 55′.6. **9.** A, B, C: 130° 0′.1, 36° 54′.8, 59° 4′.4. **11.** b, A, C: 78° 11′.1, 13° 51′.3, 129° 41′.6. **13.** b, A, B: 84° 54′, 108° 27′.8, 84° 37′.5.

Article 96 — Page 108

1. A, B, C: 81° 38′.7, 70° 9′.8, 64° 47′. **3.** A; B, C: 140° 0′, 61° 40′, 26° 30′. **5.** b, A, C: 117° 5′.4, 65° 29′.6, 123° 21′. **7.** a, B, C: 82° 7′, 111° 32′.6, 92° 28′.4. **9.** b, c, A: 157° 40′, 33° 20′, 62° 51′. **11.** b, c, A: 134° 55′.3, 62° 47′.7, 111° 39′.6. **13.** a, b, c: 163° 34′, 169° 40′, 8° 11′.6. **15.** a, b, c: 49° 24′, 149° 34′.4, 148° 33′.5. **17.** a, A, C: 118° 20′, 136° 57′, 29° 40′. **19.** c, A, C: 153° 38′.7, 42° 37′.3, 160° 1′.4; or c', A', C': 90° 5′.7, 137° 22′.7, 50° 18′.9. **21.** b, c, B: 124° 59′.4, 33° 22′, 83° 25′.6. **23.** b, c, C: 57° 35′, 154° 15′.5, 151° 15′; or b', c', C': 122° 25′, 64° 2′.2, 84° 41′.7.

Article 97 — Page 109

1. N.Y. — S.F. 2568 mi. N.Y. — M.C. 2090 mi. S.F. — M.C. 1889 mi. Angles: N.Y. 48° 58′, S.F. 55° 48′, M.C. 82° 40′. Area: 2025300 sq. mi. **3.** 9260 mi. **5.** $\lambda = 9^h\,34^m\,15^s$, $\phi = 22°\,6'$ N.; course, S. 44° 28′ W. **7.** 8° 20′; 99.5 mi. (7-place tables.)

Article 98 — Page 110

1. $A = \pm\ 92°\,50'$; $t = \pm 5^h\,4^m\,12^s$. **3.** $h = 43°\,27'$; $A = 70°\,3'$. **5.** N. 1° 33′.6 E. or W.; $t = \pm 5^h\,55^m\,54^s$. **7.** 9° 46′.4.

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APPENDIX A

The Greek Alphabet

Letters.	Name.	Letters.	Name.	Letters.	Name.
Α, α,	Alpha	Ι, ι,	Iota	Ρ, ρ,	Rho
Β, β,	Beta	Κ, κ,	Kappa	Σ, σ,	Sigma
Γ, γ,	Gamma	Λ, λ,	Lambda	Τ, τ,	Tau
Δ, δ,	Delta	Μ, μ,	Mu	Υ, υ,	Upsilon
Ε, ε,	Epsilon	Ν, ν,	Nu	Φ, φ,	Phi
Ζ, ζ,	Zeta	Ξ, ξ,	Xi	Χ, χ,	Chi
Η, η,	Eta	Ο, ο,	Omicron	Ψ, ψ,	Psi
Θ, θ, ϑ,	Theta	Π, π,	Pi	Ω, ω,	Omega

Formulas of Plane Trigonometry

Definitions. — In right triangle ABC, whose sides are a, b, c **(2)**,

$$\sin A = \frac{a}{c}, \quad \cos A = \frac{b}{c}, \quad \tan A = \frac{a}{b},$$

$$\csc A = \frac{c}{a}, \quad \sec A = \frac{c}{b}, \quad \cot A = \frac{b}{a}.$$

$$\operatorname{vers} A = 1 - \cos A. \quad \operatorname{covers} A = 1 - \sin A.$$

More generally, if x be an angle of any magnitude, as XOP in the figure of **(10)**,

$$\sin x = \frac{\text{ordinate}}{\text{distance}}, \quad \cos x = \frac{\text{abscissa}}{\text{distance}}, \quad \tan x = \frac{\text{ordinate}}{\text{abscissa}},$$

$$\csc x = \frac{\text{distance}}{\text{ordinate}}, \quad \sec x = \frac{\text{distance}}{\text{abscissa}}, \quad \cot x = \frac{\text{abscissa}}{\text{ordinate}}.$$

Relations between the Functions of an Angle. Formulas, Group A. (16.)

1. $\sin x = \dfrac{1}{\csc x}.$ 2. $\cos x = \dfrac{1}{\sec x}.$ 3. $\tan x = \dfrac{1}{\cot x}.$

4. $\tan x = \frac{\sin x}{\cos x}\cdot$ 5. $\cot x = \frac{\cos x}{\sin x}\cdot$ 6. $\sin^2 x + \cos^2 x = 1.$

7. $1 + \tan^2 x = \sec^2 x.$ 8. $1 + \cot^2 x = \csc^2 x.$

Rules for expressing any function of any angle in terms of a function of an acute angle. (17.)

Any function of any angle x is numerically equal to the $\left\{\begin{matrix}\text{same function}\\ \text{co-function}\end{matrix}\right.$ of x increased or diminished by any $\left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right.$ multiple of 90°.

The sign of the result must be determined according to the quadrant of x.

Functions of $+x$ and $-x$. (18.)

$f(+x) = f(-x)$, when f = cosine or secant.

$f(+x) = -f(-x)$, when f = sine, cosecant, tangent, cotangent.

Angles Corresponding to a Given Function. (25.)

Let θ denote the smallest positive angle having a given function equal to a given number a. Then all angles such that

I. $\left\{\begin{matrix}\sin x = a\\ \csc x = a\end{matrix}\right.$ are $x = 2n\pi + \theta$ and $(2n+1)\pi - \theta;$

II. $\left\{\begin{matrix}\cos x = a\\ \sec x = a\end{matrix}\right.$ are $x = 2n\pi \pm \theta;$

III. $\left\{\begin{matrix}\tan x = a\\ \cot x = a\end{matrix}\right.$ are $x = n\pi + \theta.$

Formulas, Group B. (44.)

9. $\sin(x+y) = \sin x \cos y + \cos x \sin y.$

10. $\cos(x+y) = \cos x \cos y - \sin x \sin y.$

11. $\sin(x-y) = \sin x \cos y - \cos x \sin y.$

12. $\cos(x-y) = \cos x \cos y + \sin x \sin y.$

13. $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}\cdot$

14. $\cot(x+y) = \frac{\cot x \cot y - 1}{\cot x + \cot y}\cdot$

15. $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}\cdot$

16. $\cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}\cdot$

Formulas, Group C. (**46.**)

Double Angle.

17. $\sin 2x = 2 \sin x \cos x.$

18. $$\cos 2x = \cos^2 x - \sin^2 x,$$
$$= 1 - 2\sin^2 x,$$
$$= 2\cos^2 x - 1.$$

19. $$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}.$$

Half-Angle.

20. $$\sin \tfrac{1}{2} x = \pm\sqrt{\frac{1 - \cos x}{2}}.$$

21. $$\cos \tfrac{1}{2} x = \pm\sqrt{\frac{1 + \cos x}{2}}.$$

22. $$\tan \tfrac{1}{2} x = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}},$$
$$= \frac{1 - \cos x}{\sin x},$$
$$= \frac{\sin x}{1 + \cos x}.$$

Formulas, Group D. (**47.**)

23. $$\sin u + \sin v = 2 \sin \frac{u+v}{2} \cos \frac{u-v}{2}.$$

24. $$\sin u - \sin v = 2 \cos \frac{u+v}{2} \sin \frac{u-v}{2}.$$

25. $$\cos u + \cos v = 2 \cos \frac{u+v}{2} \cos \frac{u-v}{2}.$$

26. $$\cos u - \cos v = -2 \sin \frac{u+v}{2} \sin \frac{u-v}{2}.$$

Right Triangles. — Solve by means of the definitions of the trigonometric functions.

Oblique Plane Triangles.

1. *Law of Sines:* $a : b : c = \sin A : \sin B : \sin C$ (**58**)
2. *Law of Cosines:* $a^2 = b^2 + c^2 - 2bc \cos A.$ (**59**)
3. *Law of Tangents:* $\dfrac{a - b}{a + b} = \dfrac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}.$ (**60**)

Half-Angles. (**61.**)

Let $s = \frac{1}{2}(a + b + c)$ and $r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}.$

4. $$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

5. $$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}.$$

6. $$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

7. $$\tan \tfrac{1}{2} A = \frac{r}{s - a}.$$

Solution of Oblique Plane Triangles.

Case I. Given two angles and a side. **(63)**
Use law of sines.

Case II. Given two sides and the included angle. **(64)**
Use law of tangents, then law of sines.

Case III. Given two sides and an opposite angle. **(65)**
Use law of sines. Ambiguous case.

Case IV. Given the three sides. **(66)**
Use one of the formulas (4), (5), (6), or (7) above, preferably the last one.

$$\text{Area} = \tfrac{1}{2} ab \sin C = \sqrt{s(s-a)(s-b)(s-c)}. \qquad \textbf{(67)}$$

FORMULAS OF SPHERICAL TRIGONOMETRY

Spherical Right Triangle. (83–86.) — Let A, B, C be the angles, and a, b, c the sides. Arrange the five parts a, b, co-B, co-c, co-A in circular order. These parts are then connected by Napier's Rules:

$$\text{sine of middle part} = \begin{cases} \text{product of cosines of opposite parts;} \\ \text{product of tangents of adjacent parts.} \end{cases}$$

To solve a spherical right triangle use Napier's Rules to write a formula involving the two given parts and a required part.

To solve a quadrantal triangle, solve its polar right triangle.

Spherical Oblique Triangles.

1. *Law of Sines:* $\sin a : \sin b : \sin c = \sin A : \sin B : \sin C.$ **(87)**
2. *Law of Cosines:* $\cos a = \cos b \cos c + \sin b \sin c \cos A.$ **(87)**
3. $\cos A = -\cos B \cos C + \sin B \sin C \cos a.$ **(88)**

Half-Angles. (89.)

$$s = \tfrac{1}{2}(a+b+c); \quad \tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.$$

5. $$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}.$$

6. $$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}.$$

7. $$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}.$$

9. $$\tan \tfrac{1}{2} A = \frac{\tan r}{\sin(s-a)}.$$

Half-Sides. (90.)

$$S = \tfrac{1}{2}(A+B+C);\quad \tan R = \sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}}.$$

14. $$\sin\tfrac{1}{2}a = \sqrt{\frac{-\cos S\cos(S-A)}{\sin B\sin C}}.$$

15. $$\cos\tfrac{1}{2}a = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}}.$$

16. $$\tan\tfrac{1}{2}a = \sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}.$$

17. $$\tan\tfrac{1}{2}a = \tan R\cos(S-A).$$

Napier's Analogies. (91.)

20′. $$\tan\tfrac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c.$$

21′. $$\tan\tfrac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c.$$

22′. $$\tan\tfrac{1}{2}(A-B) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C.$$

23′. $$\tan\tfrac{1}{2}(A+B) = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C.$$

Spherical Excess. Area. (92.)

$$E = (A+B+C) - 180^\circ.$$

$$\tan\tfrac{1}{2}E = \frac{\tan\frac{1}{2}a\tan\frac{1}{2}b\sin C}{1+\tan\frac{1}{2}a\tan\frac{1}{2}b\cos C}.$$

$$\tan\tfrac{1}{4}E = \sqrt{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)}.$$

$$\text{Area} = \frac{E\text{ (degrees)}}{720}\times 4\pi R^2 = E\text{ (radians)}\times R^2.$$

Solution of Spherical Oblique Triangle. (93.)

I. Given two sides and an opposite angle.
Use law of sines, then Napier's Analogies. Two solutions possible.

II. Given two angles and an opposite side.
As in I.

III. Given the three sides.
Use formulas for the half-angles.

IV. Given the three angles.
Use formulas for the half-sides.

V. Given two sides and their included angle.
Use Napier's Analogies, then law of sines.

VI. Given two angles and their included side.
As in V.

APPENDIX B

Explanation of the Tables and Their Use

TABLE I

Common Logarithms. Definition. *The common logarithm of a number is the exponent which must be applied to* 10 *to produce the given number.*

The symbol for the common logarithm of a number n is $\log_{10} n$, which is read:

"*The logarithm of n to the base* 10."

Examples.

$10^2 = 100$		$100 = 2,$	or	$\log_{10} 100 = 2.$
$10^3 = 1000$	∴ common	$1000 = 3,$	or	$\log_{10} 1000 = 3.$
$10^0 = 1$	logarithm	$1 = 0,$	or	$\log_{10} 1 = 0.$
$10^{-1} = 0.1$	of	$0.1 = -1,$	or	$\log_{10} 0.1 = -1.$
$10^{-2} = 0.01$		$0.01 = -2,$	or	$\log_{10} 0.01 = -2.$

In these equations 10 is called the **base** of the system of logarithms. Other numbers might be used as bases, but for purposes of computation the base in common use is 10.

In general, if $n = 10^x$,
then *the common logarithm of* $n = x$, or $\log_{10} n = x$.

Theory of Logarithms. So much of the theory of logarithms as is required in ordinary computation may be summed up in the following rules:

I. *The logarithm of a product equals the sum of the logarithms of the factors.*

$$\log_{10} m \cdot n = \log_{10} m + \log_{10} n.$$

II. *The logarithm of a fraction equals the logarithm of the numerator minus the logarithm of the denominator.*

$$\log_{10} \frac{m}{n} = \log_{10} m - \log_{10} n.$$

III. *The logarithm of the pth power of a number equals p times the logarithm of the number.*

$$\log_{10} m^p = p \log_{10} m.$$

Proofs.

I. Given two numbers m and n whose common logarithms are x and y respectively.

That is $\log_{10} m = x$ and $\log_{10} n = y$.

Then by definition of logarithms,

$$m = 10^x \quad \text{and} \quad n = 10^y.$$

Hence $m \cdot n = 10^x \cdot 10^y = 10^{x+y}.$

Therefore $\log_{10} m \cdot n = x + y = \log_{10} m + \log_{10} n.$

II. Proceeding as in I except that we divide m by n, we have

$$\frac{m}{n} = \frac{10^x}{10^y} = 10^{x-y}.$$

Therefore $\log_{10} \frac{m}{n} = x - y = \log_{10} m - \log_{10} n.$

III. To prove that $\log_{10} m^p = p \log_{10} m$, let x be the common logarithm of m.

That is $\log_{10} m = x.$

Then $m = 10^x.$

Raising to pth power: $m^p = (10^x)^p = 10^{px}.$

Therefore, by definition of a logarithm,

$$\log_{10} m^p = px = p \log_{10} m.$$

This proof holds whether p is an integer or not. In applying the formula roots are to be written as fractional exponents, thus:

$$\log_{10} \sqrt[3]{m^2} = \log_{10} m^{\frac{2}{3}} = \tfrac{2}{3} \log_{10} m.$$

Exercises. Prove:

1. $\log_{10} mnr = \log_{10} m + \log_{10} n + \log_{10} r.$
2. $\log_{10} \frac{mn}{rs} = \log_{10} m + \log_{10} n - \log_{10} r - \log_{10} s.$
3. $\log_{10} m^p n^q = p \log_{10} m + q \log_{10} n.$
4. $\log_{10} \frac{m^p}{n^q} = p \log_{10} m - q \log_{10} n.$
5. $\log_{10} \sqrt{m^3 n^5} = \frac{3}{2} \log_{10} m + \frac{5}{2} \log_{10} n.$
6. $\log_{10} \sqrt[3]{\frac{m^2 n}{r^4}} = \frac{2}{3} \log_{10} m + \frac{1}{3} \log_{10} n - \frac{4}{3} \log_{10} r.$
7. $\log_{10} \frac{(mn)^3}{\sqrt{r^5 s^3}} = 3 \log_{10} m + 3 \log_{10} n - \frac{5}{2} \log_{10} r - \frac{3}{2} \log_{10} s.$

The proofs of rules I, II, III are also valid when the base 10 is replaced by any other positive number. In what follows we deal exclusively with the base 10, and hence we shall usually omit the subscript 10, so that $\log_{10} m$ will be written merely $\log m$.

Numerous applications of these rules will be found in the explanation of the use of Table I.

Table of Common Logarithms. If we ask the question — What power of 10 will give 302? — we can see at once that the answer must lie between 2 and 3, because 302 lies between 10^2 and 10^3. That is, $302 = 10^{2.+}$, and $\log_{10} 302 = 2.+$.

The necessary decimal can be supplied by reference to a *table of logarithms*, such as Table I.

The function of such a table is to furnish the *decimal part* of the common logarithm of any number. The tables in this text give these decimals to four places. For more accurate computations 5-place, 6-place, and 7-place tables are in common use. The integral part of the logarithm is to be supplied by the computer.

Definitions. *The integral part of a logarithm is called the* **characteristic,** *and the decimal part the* **mantissa.**

Rules for Characteristics.

(**a**) When the number has n significant figures to the left of the decimal point, the characteristic of its logarithm is $n - 1$.

(**b**) When the number is a decimal with n ciphers between the decimal point and the first digit which is not zero, the characteristic of its logarithm is $9 - n$, and -10 must be supplied to complete the logarithm.

The reason for these rules will become evident when we consider an example.

Example. Let us find log 302. In the table find 30 in the left-hand column and run across the page horizontally to the column headed 2. There we find that mantissa of log 302 = .4800. Now 302 lies between 100 and 1000, i.e. between 10^2 and 10^3. Hence, by the definition of a logarithm, log 302 must lie between 2 and 3. Therefore the characteristic is 2, and

$$\log 302 = 2.4800.$$

This is of course not the *exact* logarithm of 302, but only its value to four decimal places.

Writing the last equation in exponential form, we have

$$302 = 10^{2.4800}.$$

Multiplying both sides by 10,

$$3020 = 10 \times 10^{2.4800} = 10^{3.4800}. \quad \text{Hence, } \log 3020 = 3.4800.$$

Multiplying again by 10,

$$30200 = 10 \times 10^{3.4800} = 10^{4.4800}. \quad \text{Hence, } \log 30200 = 4.4800.$$

Therefore, where a number is multiplied by 10, the characteristic of its logarithm is increased by 1; the mantissa remains unchanged.

Dividing the above equation successively by 10, we obtain

$$\begin{aligned} 30.2 &= 10^{2.4800} \div 10 = 10^{1.4800}, \\ 3.02 &= 10^{1.4800} \div 10 = 10^{0.4800}, \\ .302 &= 10^{0.4800} \div 10 = 10^{0.4800-1}, \\ .0302 &= 10^{0.4800-1} \div 10 = 10^{0.4800-2}, \\ .00302 &= 10^{0.4800-2} \div 10 = 10^{0.4800-3}, \end{aligned}$$

and so on. As logarithmic equations these are:

$$\begin{aligned} \log 30.2 &= 1.4800, \\ \log 3.02 &= 0.4800, \\ \log .302 &= 0.4800 - 1 = 9.4800 - 10, \\ \log .0302 &= 0.4800 - 2 = 8.4800 - 10, \\ \log .00302 &= 0.4800 - 3 = 7.4800 - 10, \end{aligned}$$

and so on. The second form in the last three equations is used for convenience in computations; it is in accordance with rule (**b**).

To discuss rules (**a**) and (**b**) more generally, let m be any number. Then by the definition of a logarithm, when

	m lies between	$\log m$ lies between
(1)	1 and 10,	0 and 1,
(2)	10 and 100,	1 and 2,
(3)	100 and 1000,	2 and 3,
(4)	1000 and 10000,	3 and 4,

and so on. Therefore, when m has

(1) 1 digit to the left of the point, $\log m = 0.+ \cdots$;
(2) 2 digits to the left of the point, $\log m = 1.+ \cdots$;
(3) 3 digits to the left of the point, $\log m = 2.+ \cdots$;
(4) 4 digits to the left of the point, $\log m = 3.+ \cdots$;

and so on. Hence rule (**a**).

In the case of decimal numbers,

	when m lies between	$\log m$ lies between
(1)	1 and 0.1,	0 and -1,
(2)	0.1 and 0.01,	-1 and -2,
(3)	0.01 and 0.001,	-2 and -3,
(4)	0.001 and 0.0001,	-3 and -4,

and so on. That is, when m is a decimal number in which

(1) no cipher follows the point, $\log m = 9.+ \cdots - 10$;
(2) 1 cipher follows the point, $\log m = 8.+ \cdots - 10$;
(3) 2 ciphers follow the point, $\log m = 7.+ \cdots - 10$;
(4) 3 ciphers follow the point, $\log m = 6.+ \cdots - 10$;

and so on. Hence rule (**b**).

Interpolation. — *Example.* Find log 3024.
From the table,

$$\left.\begin{array}{l}\text{mantissa of } \log 302 = .4800;\\ \text{mantissa of } \log 303 = .4814;\end{array}\right. \text{difference} = .0014.$$

Assuming that the increase in the logarithm is proportional to the increase in the number, we have

$$\text{mantissa of } \log 3024 = .4800 + .4 \times .0014 = .4806.$$

The result is here given to the nearest unit in the fourth decimal place, $.4 \times .0014$ being taken equal to .0006 in place of .00056.

Proportional Parts. — For convenience in interpolation, the tabular differences greater than 20 are subdivided into tenths and tabulated under the heading "Prop. Parts." When the difference is less than 20, the interpolation is best made mentally. If it is desired, the table of proportional parts may be used when $d < 20$ by taking half the proportional part corresponding to double the difference.

Examples.

1. $\log 164.3 = ?$

$$\begin{array}{rl}\text{Mantissa of } \log 164 = & .2148; \quad d = 27,\\ \text{Correction for } .3 = & \underline{\qquad\ \ 8}\\ \log 164.3 = & 2.2156\end{array}$$

2. $\log (164.3)^{\frac{2}{3}} = ?$

$$\begin{aligned}\log (164.3)^{\frac{2}{3}} &= \tfrac{2}{3} \log 164.3,\\ &= \tfrac{2}{3}(2.2156) = 1.4771.\end{aligned}$$

3. $\log .01047 = ?$

$$\begin{aligned} \text{Mantissa of } \log 104 &= .0170; \qquad d = 42, \\ \text{Correction for } .7 &= \underline{29} \\ \log .01047 &= 8.0199 - 10 \end{aligned}$$

4. $\log \sqrt[3]{(.01047)^4} = ?$

$$\begin{aligned} \sqrt[3]{.01047^4} &= (.01047)^{\frac{4}{3}}, \\ \log \sqrt[3]{(.01047)^4} &= \tfrac{4}{3} \log (.01047), \\ &= \tfrac{4}{3} (8.0199 - 10). \\ 4\,(8.0199 - 10) &= 32.0796 - 40 = 22.0796 - 30. \\ \tfrac{1}{3}\,(22.0796 - 30) &= 7.3599 - 10. \end{aligned}$$

Note. When a logarithm which is followed by -10 is to be divided by a number, add and subtract a multiple of ten so that the quotient will come out in a form followed by -10. Thus:

$$\tfrac{1}{4}\,(8.2448 - 10) = \tfrac{1}{4}\,(38.2448 - 40) = 9.5612 - 10.$$

Anti-logarithm. — The number whose logarithm is x is called the *anti-logarithm* of x.

Thus, if $x = \log m$, then $m = \text{anti-log}\, x$.

Given a logarithm, to obtain the corresponding number (anti-logarithm).

Examples.

1. $\log m = 0.4806. \quad m = ?$

The given logarithm lies between the tabular logarithms .4800 and .4814, to which correspond the numbers 302 and 303 respectively. Thus we have

$$\begin{array}{cl} \text{Number.} & \text{Mantissa of log.} \\ 302 & \left.\begin{array}{l} .4800 \\ .4806 \end{array}\right\} 6 \\ m & \\ 303 & .4814 \end{array} \quad \left.\vphantom{\begin{array}{l} . \\ . \\ . \end{array}}\right\} 14$$

Hence, without regard to the decimal point, $m = 302 + \frac{6}{14} = 3024+$.
Pointing off properly,

$$m = \text{anti-log}\, 0.4806 = 3.024+.$$

2. $\log m = 7.0959 - 10. \quad m = ?$

$$\begin{aligned} \text{mantissa of } \log 124 &= .0934 \\ \text{mantissa of } \log\ m &= .0959 \\ \text{mantissa of } \log 125 &= .0969 \end{aligned} \quad \begin{array}{l} \left.\vphantom{\begin{array}{l} . \\ . \end{array}}\right\} 25 \\ \end{array} \left.\vphantom{\begin{array}{l} . \\ . \\ . \end{array}}\right\} 35$$

Hence m has the sequence of figures

$$124 + \tfrac{25}{35} = 1247 +.$$

Pointing off properly,

$$m = \text{anti-log}\,(7.0959 - 10) = .001247+.$$

Note. The value of the quotient $\frac{25}{35}$ may be obtained from the column of Prop. Parts by finding the number of tenths of 35 required to equal 25. We have from this column,

$$.7 \times 35 = 24.5 \text{ and } .8 \times 35 = 28.0.$$

Hence we see that to make 25 we need a little more than $.7 \times 35$. A close approximation would be $.71+$, making $m = .0012471+$.

When the tabular difference is large, it is possible to obtain correctly more than four significant figures of a number when its four-place logarithm is given.

Cologarithm. — The *cologarithm* of a number is the logarithm of the reciprocal of the number.

Thus: $\text{colog}\, m = \log\frac{1}{m} = \log 1 - \log m = -\log m.$

In practice we usually write it in the form

$$\text{colog}\, m = -\log m = (10 - \log m) - 10.$$

Rule. To form the cologarithm of a number, subtract its logarithm from 10 and write -10 after the result.

Examples.

1. $\text{colog}\, 302 = (10 - \log 302) - 10$
$= (10 - 2.4800) - 10 = 7.5200 - 10.$

2. $\text{colog}\, .003024 = (10 - \log .003024) - 10$
$= (10 - [7.4806 - 10]) - 10 = 2.5194.$

Use of the Cologarithm.

Example. Calculate the value of $\frac{302 \times .415}{541 \times .0828}$.

Let m be the value of the given fraction. Then without the use of cologarithms the calculation is as follows.

$$\log m = \log 302 + \log .415 - \log 541 - \log .0828.$$

$$\begin{array}{rlrl} \log 302 = & 2.4800 & \log 541 = & 2.7332 \\ \log .415 = & \underline{9.6180 - 10} & \log .0828 = & \underline{8.9180 - 10} \\ & 12.0980 - 10 & & 11.6512 - 10 \\ & \underline{11.6512 - 10} & & \\ \log m = & 0.4468, & m = & 2.7975. \end{array}$$

To use cologarithms, we write

$$m = 302 \times .415 \times \frac{1}{541} \times \frac{1}{.0828}.$$

$$\log m = \log 302 + \log .415 + \text{colog}\, 541 + \text{colog}\, .0828$$

$$\begin{array}{rl} \log 302 = & 2.4800 \\ \log .415 = & 9.6180 - 10 \\ \text{colog}\, 541 = & 7.2668 - 10 \\ \text{colog}\, .0828 = & \underline{1.0820 \qquad\;\;} \\ \log m = & 20.4468 - 20 \\ m = & 2.7975. \end{array}$$

As a last example, we calculate the value of the quantity,

$$m = \sqrt{\frac{(.00812)^{\frac{2}{3}} \times (-471.2)^3}{(-522.3)^3 \times (.01242)^{\frac{3}{4}}}}.$$

To take account of the signs, which must be done independently of the logarithmic calculation, we note that the cube of a negative quantity occurs on both sides of the fraction; hence the sign of the fraction is plus.

We now write

$$\log m = \tfrac{1}{2}\left[\log (.00812)^{\frac{2}{3}} + \log (471.2)^3 + \operatorname{colog} (522.3)^3 + \operatorname{colog} (.01242)^{\frac{3}{4}}\right].$$

$$\begin{aligned}
\log .00812 &= 7.9096 - 10 & \qquad \log (.00812)^{\frac{2}{3}} &= 8.6064 - 10\\
\log 471.2 &= 2.6732 & \log (471.2)^3 &= 8.0196\\
\log 522.3 &= 2.7179 & \log (522.3)^3 &= 8.1537\\
\log .01242 &= 8.0941 - 10 & \log (.01242)^{\frac{3}{4}} &= 8.5706 - 10
\end{aligned}$$

Hence

$$\begin{aligned}
\log (.00812)^{\frac{2}{3}} &= 8.6064 - 10\\
\log (471.2)^3 &= 8.0196\\
\operatorname{colog} (522.3)^3 &= 1.8463 - 10\\
\operatorname{colog} (.01242)^{\frac{3}{4}} &= 1.4294\\
2\,&\underline{|19.9017 - 20}\\
\log m &= 9.9508 - 10\\
m &= .8929.
\end{aligned}$$

Exercises. Verify the following equations:

1. $\log 7 = 0.8451.$
2. $\log 253 = 2.4031.$
3. $\log 253.5 = 2.4040.$
4. $\log .0253 = 8.4031 - 10.$
5. $\log .002533 = 7.4036 - 10$
6. $\log 6544 = 3.8158.$
7. $\log 4.007 = 0.6028.$
8. $\log .9995 = 9.9998 - 10.$
9. $\log \sqrt{766} = 1.4421.$
10. $\log \frac{1}{766} = 7.1158 - 10.$
11. $\log (.0022)^3 = 2.0272 - 10.$
12. $\log \sqrt[3]{.0022} = 9.1141 - 10.$
13. $\log (.01401)^{\frac{4}{5}} = 8.5171 - 10.$
14. $\log (.0003684)^{\frac{7}{2}} = 7.9820 - 20.$
15. $\operatorname{colog} 200 = 7.6990 - 10.$
16. $\operatorname{colog} .7 = 0.1549.$
17. $\operatorname{colog} .0448 = 1.3487.$
18. $\operatorname{colog} \sqrt{5475} = 8.1308 - 10.$

19. colog $(.0003684)^{\frac{7}{2}} = 12.0180.$

20. antilog $1.2222 = 16.68.$

21. antilog $3.6675 = 4650.$

22. antilog $0.4000 = 2.5118.$

23. antilog $(8.3250 - 10) = .021135.$

24. antilog $(6.9525 - 10) = .0008964.$

25. $(.748)^3 = .4185.$

26. $\sqrt[3]{-.0822} = -.4348.$

27. $(-6.213)^{\frac{2}{5}} = 2.076.$

28. $\dfrac{(-.1412)^2}{\sqrt[3]{-(.00475)}} = -.11858.$

29. $\dfrac{1}{(72.32)^{\frac{2}{3}}} = .05761.$

TABLE II.

This table gives the logarithms of the sine, cosine, tangent and cotangent of angles from 0° to 90°, at intervals of 10′.

When the angle is taken from the left-hand column of the page, the name of the function must be sought at the top of the page; when the angle is taken from the right-hand column of the page, the name of the function must be sought at the foot of the page.

When the function is numerically less than 1, -10 must be written after its tabular logarithm. This is the case with the sines and cosines of all angles between 0° and 90°, with tangents of angles between 0° and 45°, and with cotangents between 45° and 90°.

For convenience in interpolation the differences of the tabular logarithms are given, and these differences are subdivided into tenths in the column of proportional parts. Hence this column contains the corrections to the tabular logarithms for each minute of angle from 1′ to 9′ inclusive. These corrections are to be added when the logarithm increases with the angle, and they are to be subtracted when the logarithm decreases as the angle increases.

When the logarithm of a function of an angle greater than 90° is required, change to the equivalent function of an angle less than 90° (**17**). Algebraic signs must be adjusted independently of the logarithmic calculation, as in the use of Table I.

Seconds of arc must be reduced to the equivalent fractions of a minute of arc.

To obtain log sec x, take from the table colog cos x; for log csc x use colog sin x.

Examples.

1. log sin 20° 13′ = ?

$$\begin{aligned} \log\sin 20^\circ\,10' &= 9.5375; \qquad d = 34. \\ d \text{ for } 3' \text{ (Prop. Parts)} &= \underline{10.2} \\ \log\sin 20^\circ\,13' &= 9.5385 - 10. \end{aligned}$$

2. $\log\cos 20^\circ\,13' = ?$

$$\begin{aligned}\log\cos 20^\circ\,10' &= 9.9725; \qquad d = 4.\\ d \text{ for } 3' = 4 \times .3 &= \underline{\quad 1.2}\\ \log\cos 20^\circ\,13' &= 9.9724 - 10.\end{aligned}$$

3. $\log\tan 29^\circ\,47' = ?$

$$\begin{aligned}\log\tan 29^\circ\,40' &= 9.7556; \qquad d = 29.\\ d \text{ for } 7' \text{ (Prop. Parts)} &= \underline{\quad 20.3}\\ \log\tan 29^\circ\,47' &= 9.7576 - 10\end{aligned}$$

The same result may also be obtained by starting with log tan 29° 50′, thus:

$$\begin{aligned}\log\tan 29^\circ\,50' &= 9.7585; \qquad d = 29.\\ d \text{ for } 3' &= \underline{\quad 8.7}\\ \log\tan 29^\circ\,47' &= 9.7576 - 10.\end{aligned}$$

As a rule, in interpolating start from the nearest tabular number.

4. $\log\cot 29^\circ\,47' = ?$

$$\begin{aligned}\log\cot 29^\circ\,50' &= 0.2415; \qquad d = 29.\\ d \text{ for } 3' &= \underline{\quad 8.7}\\ \log\cot 29^\circ\,47' &= 0.2424.\end{aligned}$$

5. $\log\sin 58^\circ\,44' = ?$

$$\begin{aligned}\log\sin 58^\circ\,40' &= 9.9315; \qquad d = 8.\\ d \text{ for } 4' &= \underline{\quad 3.2}\\ \log\sin 58^\circ\,44' &= 9.9318 - 10.\end{aligned}$$

6. $\log\tan 67^\circ\,23'.5 = ?$

$$\begin{aligned}\log\tan 67^\circ\,20' &= 0.3792; \qquad d = 36.\\ d \text{ for } 3'.5 = 10.8 + 1.8 &= \underline{\quad 12.6}\\ \log\tan 67^\circ\,23'.5 &= 0.3805.\end{aligned}$$

Here we obtain d for 3′.5 from d for 3′ + d for 0′.5. Note that d for 0.5 is simply one-tenth of d for 5′.

7. $\log\cos 105^\circ\,51'.6 = ?$

$$\cos 105^\circ\,51'.6 = -\sin 15^\circ\,51'.6.$$

Neglecting the algebraic sign we have

$$\begin{aligned}\log\sin 15^\circ\,50' &= 9.4359; \qquad d = 44.\\ d \text{ for } 1'.6 &= \underline{\quad 7.0}\\ \log\sin 15^\circ\,51'.6 &= 9.4366 - 10 = \log\cos 105^\circ\,51'.6.\end{aligned}$$

8. $\log\tan 250^\circ\,34'.3 = ?$

$$\begin{aligned}\tan 250^\circ\,34'.3 &= \tan 70^\circ\,34'.3.\\ \log\tan 70^\circ\,30' &= 0.4509; \qquad d = 40.\\ d \text{ for } 4'.3 &= \underline{\quad 17.2}\\ \log\tan 70^\circ\,34'.3 &= 0.4526 = \log\tan 250^\circ\,34'.3.\end{aligned}$$

Angles near 0° or near 90°.

When an angle, x, lies near 0°, $\sin x$, $\tan x$, and $\cot x$ vary too rapidly with x to permit of accurate interpolation of their logarithms from the table. The same is true of $\cos x$, $\tan x$, and $\cot x$, when x lies near 90°. We will show how accurate values of these logarithms may be obtained.

Let $$S = \log \frac{\sin x}{x} \quad \text{and} \quad T = \log \frac{\tan x}{x},$$

x being expressed in minutes of arc.

Then $$\log \sin x = \log x' + S,$$

and $$\log \tan x = \log x' + T.$$

When x is small the quantities S and T vary quite slowly with x. The values of S and T are given in the last column of the first page of Table II, x ranging from 0° to 5°; -10 is to be added to the tabular numbers there given.

To get $\log \sin x$, reduce x to minutes of arc and take $\log x'$ from Table I; to this logarithm add S.

To get $\log \tan x$, add T to $\log x'$.

To get $\log \cot x$, first get $\log \tan x$ and form the cologarithm of the result.

For, $$\log \cot x = \operatorname{colog} \tan x.$$

To obtain $\log \cos x$, $\log \tan x$ or $\log \cot x$, when x lies between 85° and 90°, calculate the co-function of the complementary angle by the method given above.

To find the angle from $\log \sin x$, $\log \tan x$ or $\log \cot x$, when x lies near 0°, we use the relations

$$\log x' = \log \sin x - S;$$
$$\log x' = \log \tan x - T;$$
$$\log x' = -\log \cot x - T.$$

The necessary values of S and T can be obtained after finding an approximate value of x from Table II.

To find x from $\log \cos x$, $\log \tan x$, or $\log \cot x$, when x lies near 90°, replace

$$\log \cos x \quad \text{by} \quad \log \sin (90° - x);$$
$$\log \tan x \quad \text{by} \quad \log \cot (90° - x);$$
$$\log \cot x \quad \text{by} \quad \log \tan (90° - x).$$

Then $90° - x$ can be obtained by the method given above for angles near $0°$. Hence x is determined.

Examples.

1. Find $\log \sin x$, $\log \tan x$ and $\log \cot x$ when $x = 1° 22' 12''$.

$$\begin{array}{rlrl} x = 1° 22' 12'' = & 82'.2. & \log x' = & \log 82.2 = 1.9149. \\ \log x = & 1.9149 & \log x = & 1.9149 \\ S = & \underline{6.4637 - 10} & T = & \underline{6.4638 - 10} \\ \log \sin x = & 8.3786 - 10 & \log \tan x = & 8.3787 - 10 \end{array}$$

$$\log \cot x = \operatorname{colog} \tan x = 1.6213.$$

2. Find $\log \cos x$, $\log \tan x$ and $\log \cot x$ when $x = 89° 5' 50''$.

Let $y = 90° - x = 54' 10'' = 54'.17.$

Then $\log \cos x$, $\log \tan x$, $\log \cot x$ are equal respectively to $\log \sin y$, $\log \cot y$, $\log \tan y$, which may be found as in example 1.

3. $\log \sin x = 8.2142; \quad x = ?$

From Table II, $x = 50' + ;$ hence $S = 6.4637 - 10.$

$$\begin{aligned} \log \sin x &= 8.2142 - 10 \\ S &= \underline{6.4637 - 10} \\ \log x' &= 1.7505; \qquad x = 56'.30 = 56' 18''. \end{aligned}$$

4. $\log \tan x = 8.0804 - 10; \quad x = ?$

From Table II, $x = 40' + ;$ hence $T = 6.4638$

$$\begin{aligned} \log \tan x &= 8.0804 - 10 \\ T &= \underline{6.4638 - 10} \\ \log x' &= 1.6166; \qquad x = 41'.36 = 41' 21''.6. \end{aligned}$$

5. $\log \cot x = 8.6276 - 10; \quad x = ?$

Let $y = 90° - x.$

Then $\log \tan y = \log \cot x = 8.6276 - 10.$

From Table II, $y = 2° 20' + ;$ hence $T = 6.4640.$

$$\begin{aligned} \log \tan y &= 8.6276 - 10 \\ T &= \underline{6.4640 - 10} \\ \log y' &= 2.1636; \quad y = 145'.73 = 2° 25' 44''. \end{aligned}$$

Hence $x = 90° - y = 87° 34' 16''.$

Let the student obtain the results required in the last five examples by direct interpolation from Table II.

Exercises. Verify the following equations:

1. $\log \sin 20° 40' = 9.5477 - 10.$
2. $\log \cos 66° 30' = 9.6007 - 10.$
3. $\log \tan 29° 35' = 9.7541 - 10.$
4. $\log \cot 37° 25' = 0.1163.$
5. $\log \sec 55° 50' = 0.2506.$
6. $\log \csc 44° 50' = 0.1518.$
7. $\log \tan 63° 27' = 0.3013.$
8. $\log \sin 81° 29' = 9.9952.$
9. $\log \sin 81° 31' = 9.9952.$
10. $\log \cos 81° 29' = 9.1706 - 10.$
11. $\log \cos 81° 31' = 9.1689 - 10.$
12. $\log \cot 9° 6' = 0.7954.$
13. $\log \sin 152° 27' = 9.6651 - 10.$
14. $\log \sin 2° 10' 10'' = 8.5781 - 10.$
15. $\log \tan 1° 34' 20'' = 8.4385 - 10.$
16. $\log \cot 0° 10' 22'' = 2.5206.$
17. $\log \cos 89° 28' 44'' = 7.9588 - 10.$
18. $\log \tan 88° 46' 14'' = 1.6683.$

19. $\log\sin x = 9.7926$; $x = 38^\circ 20'$.
20. $\log\sin x = 9.3548$; $x = 13^\circ 5'$.
21. $\log\sin x = 9.8867$; $x = 50^\circ 23'$.
22. $\log\cos x = 9.6030$; $x = 66^\circ 22'$.
23. $\log\tan x = 0.6278$; $x = 77^\circ 44'.5$.
24. $\log\cot x = 0.0906$; $x = 39^\circ 4'$.
25. $\log\cot x = 0.6648$; $x = 12^\circ 12'.5$.
26. $\log\sec x = 0.1374$; $x = 43^\circ 13'$.
27. $\log\csc x = 0.2890$; $x = 30^\circ 56'$.
28. $\log\sec x = 0.6680$; $x = 77^\circ 35'.8$.
29. $\log\sin x = 8.3698$; $x = 1^\circ 20' 34''$.
30. $\log\tan x = 8.7659$; $x = 3^\circ 20' 18''$.
31. $\log\cot x = 1.2952$; $x = 2^\circ 54' 3''$.
32. $\log\cos x = 8.5387$; $x = 88^\circ 1' 8''$.
33. $\log\cot x = 7.9485$; $x = 89^\circ 29' 28''$.
34. $\log\csc x = 2.3549$; $x = 0^\circ 15' 11''$.
35. $\log\sec x = 1.5102$; $x = 88^\circ 13' 48''$.

TABLE III

This table gives the numerical values of the six trigonometric functions of angles from 0° to 90° at intervals of 10′. The functions of intermediate angles are to be obtained by interpolation.

By using the tables inversely, an angle may be found, usually to the nearest minute, when a function of the angle is known to four decimal places.

TABLE IV

This is a conversion table for changing from sexagesimal to radian measure, and conversely. The entries are given to five decimal places in radians, corresponding nearly to 2″ in sexagesimal measure.

Examples.

1. Express 200° 44′ 36″ in radian measure.

$$200^\circ = 3 \times 60^\circ + 20^\circ$$

$$\begin{aligned} 3 \times 60^\circ &= 3 \times 1.04720 = 3.14160 \text{ radians.}\\ 20^\circ &= 0.34907\\ 44' &= 0.01280\\ 36'' &= \underline{0.00017}\\ 200^\circ 44' 36'' &= 3.50364 \text{ radians.} \end{aligned}$$

2. Express 3.50364 radians in sexagesimal measure.

$$\begin{aligned} 3.0 \text{ radians} &= 171^\circ 53' 14''\\ 0.5 \quad '' \quad &= 28^\circ 38' 52''\\ 0.003 \quad '' \quad &= 10' 19''\\ 0.0006 \quad '' \quad &= 2' \ 4''\\ 0.00004 \quad '' \quad &= \underline{8''}\\ 3.50364 \text{ radians} &= 200^\circ 44' 37'' \end{aligned}$$

TABLE V

This table contains the values of a number of mathematical constants, generally to fifteen places of decimals.

TABLE VI

This table gives the values of the natural or Naperian logarithm of x, and of the ascending and descending exponential functions e^x and e^{-x}, from $x = 0$ to $x = 5$ at intervals of 0.05. As a rule the tabular entries are given to three decimal places.

TABLE VII

This table gives the values of n^2, n^3, $\sqrt{n}$, and $\sqrt[3]{n}$, for values of n from 1 to 100.

The direct use of the table requires no explanation. As an example of its inverse use we find the approximate value of $\sqrt[3]{320}$. We have

$$(6.8)^3 = 314.432 \quad (n = 68),$$
$$(6.9)^3 = 328.509 \quad (n = 69).$$

Hence, interpolating linearly,

$$(6.840)^3 = 320 \text{ approx., or } \sqrt[3]{320} = 6.840+.$$

TABLES

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

Prop. Parts		
	43	**42**
1	4.3	4.2
2	8.6	8.4
3	12.9	12.6
4	17.2	16.8
5	21.5	21.0
6	25.8	25.2
7	30.1	29.4
8	34.4	33.6
9	38.7	37.8
	41	**40**
1	4.1	4.0
2	8.2	8.0
3	12.3	12.0
4	16.4	16.0
5	20.5	20.0
6	24.6	24.0
7	28.7	28.0
8	32.8	32.0
9	36.9	36.0
	39	**38**
1	3.9	3.8
2	7.8	7.6
3	11.7	11.4
4	15.6	15.2
5	19.5	19.0
6	23.4	22.8
7	27 3	26.6
8	31.2	30.4
9	35.1	34.2
	37	**36**
1	3.7	3.6
2	7.4	7.2
3	11.1	10.8
4	14.8	14.4
5	18.5	18.0
6	22.2	21.6
7	25.9	25.2
8	29.6	28.8
9	33.3	32.4
	35	**34**
1	3.5	3.4
2	7.0	6.8
3	10.5	10.2
4	14.0	13.6
5	17.5	17.0
6	21.0	20.4
7	24.5	23.8
8	28.0	27.2
9	31.5	30.6
	33	**32**
1	3.3	3.2
2	6.6	6.4
3	9.9	9.6
6	13.2	12.8
5	16.5	16.0
6	19.8	19.2
7	23.1	22.4
8	26.4	25.6
9	29.7	28.8
Prop. Parts		

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	31	**30**
1	3.1	3.0
2	6.2	6.0
3	9.3	9.0
4	12.4	12.0
5	15.5	15.0
6	18.6	18.0
7	21.7	21.0
8	24.8	24.0
9	27.9	27.0
	29	**28**
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2
	27	**26**
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4
	25	**24**
1	2.5	2.4
2	5.0	4.8
3	7.5	7.2
4	10.0	9.6
5	12.5	12.0
6	15.0	14.4
7	17.5	16.8
8	20.0	19.2
9	22.5	21.6
	23	**22**
1	2.3	2.2
2	4.6	4.4
3	6.9	6.6
4	9.2	8.8
5	11.5	11.0
6	13.8	13.2
7	16.1	15.4
8	18.4	17.6
9	20.7	19.8
	21	
1	2.1	
2	4.2	
3	6.3	
4	8.4	
5	10.5	
6	12.6	
7	14.7	
8	16.8	
9	18.9	

Prop. Parts

To be cut off

x	log sin	d	log cos	d	log tan	d	log cot	
0° 0′	−∞		10.0000	0	−∞		∞	90° 0′
10′	7.4637	3011	.0000	0	7.4637	3011	2.5363	50′
20′	.7648	1760	.0000	0	.7648	1761	.2352	40′
30′	.9408	1250	.0000	0	.9409	1249	.0591	30′
40′	8.0658	969	.0000	0	8.0658	969	1.9342	20′
50′	.1627	792	.0000	1	.1627	792	.8373	10′
1° 0′	8.2419	669	9.9999	0	8.2419	670	1.7581	89° 0′
10′	.3088	580	.9999	0	.3089	580	.6911	50′
20′	.3668	511	.9999	0	.3669	512	.6331	40′
30′	.4179	458	9.9999	1	.4181	457	.5819	30′
40′	.4637	413	.9998	0	.4638	415	.5362	20′
50′	.5050	378	.9998	1	.5053	378	.4947	10′
2° 0′	8.5428	348	9.9997	0	8.5431	348	1.4569	88° 0′
10′	.5776	321	.9997	1	.5779	322	.4221	50′
20′	.6097	300	.9996	0	.6101	300	.3899	40′
30′	.6397	280	.9996	1	.6401	281	.3599	30′
40′	.6677	263	.9995	0	.6682	263	.3318	20′
50′	.6940	248	.9995	1	.6945	249	.3055	10′
3° 0′	8.7188	235	9.9994	1	8.7194	235	1.2806	87° 0′
10′	.7423	222	.9993	0	.7429	223	.2571	50′
20′	.7645	212	.9993	1	.7652	213	.2348	40′
30′	.7857	202	.9992	1	.7865	202	.2135	30′
40′	.8059	192	.9991	1	.8067	194	.1933	20′
50′	.8251	185	.9990	1	.8261	185	.1739	10′
4° 0′	8.8436	177	9.9989	0	8.8446	178	1.1554	86° 0′
10′	.8613	170	.9989	1	.8624	171	.1376	50′
20′	.8783	163	.9988	1	.8795	165	.1205	40′
30′	.8946	158	.9987	1	.8960	158	.1040	30′
40′	.9104	152	.9986	1	.9118	154	.0882	20′
50′	.9256	147	.9985	2	.9272	148	.0728	10′
5° 0′	8.9403	142	9.9983	1	8.9420	143	1.0580	85° 0′
10′	.9545	137	.9982	1	.9563	138	.0437	50′
20′	.9682	134	.9981	1	.9701	135	.0299	40′
30′	.9816	129	.9980	1	.9836	130	.0164	30′
40′	.9945	125	.9979	2	.9966	127	.0034	20′
50′	9.0070	122	.9977	1	9.0093	123	0.9907	10′
6° 0′	9.0192	119	9.9976	1	9.0216	120	0.9784	84° 0′
10′	.0311	115	.9975	2	.0336	117	.9664	50′
20′	.0426	113	.9973	1	.0453	114	.9547	40′
30′	.0539	109	.9972	1	.0567	111	.9433	30′
40′	.0648	107	.9971	2	.0678	108	.9322	20′
50′	.0755	104	.9969	1	.0786	105	.9214	10′
7° 0′	9.0859	102	9.9968	2	9.0891	104	0.9109	83° 0′
10′	.0961	99	.9966	2	.0995	101	.9005	50′
20′	.1060	97	.9964	1	.1096	98	.8904	40′
30′	.1157		.9963		.1194		.8806	30′
	log cos	d	log sin	d	log cot	d	log tan	*x*

Small Angles

x	S	T
<1°	6.4637	6.4637
1°	6.4637	6.4638
2°	6.4636	6.4639
3°	6.4635	6.4641
4°	6.4634	6.4644
5°	6.4631	6.4649

Prop. Parts.

	113	111	109
1	11.3	11.1	10.9
2	22.6	22.2	21.8
3	33.9	33.3	32.7
4	45.2	44.4	43.6
5	56.5	55.5	54.5
6	67.8	66.6	65.4
7	79.1	77.7	76.3
8	90.4	88.8	87.2
9	101.7	99.9	98.1

	108	107	105
1	10.8	10.7	10.5
2	21.6	21.4	21.0
3	32.4	32.1	31.5
4	43.2	42.8	42.0
5	54.0	53.5	52.5
6	64.8	64.2	63.0
7	75.6	74.9	73.5
8	86.4	85.6	84.0
9	97.2	96.3	94.5

	104	102	101
1	10.4	10.2	10.1
2	20.8	20.4	20.2
3	31.2	30.6	30.3
4	41.6	40.8	40.4
5	52.0	51.0	50.5
6	62.4	61.2	60.6
7	72.8	71.4	70.7
8	83.2	81.6	80.8
9	93.6	91.8	90.9

	99	98	97	95
1	9.9	9.8	9.7	9.5
2	19.8	19.6	19.4	19.0
3	29.7	29.4	29.1	28.5
4	39.6	39.2	38.8	38.0
5	49.5	49.0	48.5	47.5
6	59.4	58.8	58.2	57.0
7	69.3	68.6	67.9	66.5
8	79.2	78.4	77.6	76.0
9	89.1	88.2	87.3	85.5

	143	142	138	137	135	134	130	129	127	125	123	122	119	117	115	114
1	14.3	14.2	13.8	13.7	13.5	13.4	13.0	12.9	12.7	12.5	12.3	12.2	11.9	11.7	11.5	11.4
2	28.6	28.4	27.6	27.4	27.0	26.8	26.0	25.8	25.4	25.0	24.6	24.4	23.8	23.4	23.0	22.8
3	42.9	42.6	41.4	41.1	40.5	40.2	39.0	38.7	38.1	37.5	36.9	36.6	35.7	35.1	34.5	34.2
4	57.2	56.8	55.2	54.8	54.0	53.6	52.0	51.6	50.8	50.0	49.2	48.8	47.6	46.8	46.0	45.6
5	71.5	71.0	69.0	68.5	67.5	67.0	65.0	64.5	63.5	62.5	61.5	61.0	59.5	58.5	57.5	57.0
6	85.8	85.2	82.8	82.2	81.0	80.4	78.0	77.4	76.2	75.0	73.8	73.2	71.4	70.2	69.0	68.4
7	100.1	99.4	96.6	95.9	94.5	93.8	91.0	90.3	88.9	87.5	86.1	85.4	83.3	81.9	80.5	79.8
8	114.4	113.6	110.4	109.6	108.0	107.2	104.0	103.2	101.6	100.0	98.4	97.6	95.2	93.6	92.0	91.2
9	128.7	127.8	124.2	123.3	121.5	120.6	117.0	116.1	114.3	112.5	110.7	109.8	107.1	105.3	103.5	102.6

Log.trig. funct's.
0° – 15°
90° – 75°

x	log sin	d	log cos	d	log tan	d	log cot	
30′	9.1157	95	9.9963	2	9.1194	97	0.8806	30′
40′	.1252	93	.9961	2	.1291	94	.8709	20′
50′	.1345	91	.9959	1	.1385	93	.8615	10′
8° 0′	9.1436	89	9.9958	2	9.1478	91	0.8522	**82° 0′**
10′	.1525	87	.9956	2	.1569	89	.8431	50′
20′	.1612	85	.9954	2	.1658	87	.8342	40′
30′	.1697	84	.9952	2	.1745	86	.8255	30′
40′	.1781	82	.9950	2	.1831	84	.8169	20′
50′	.1863	80	.9948	2	.1915	82	.8085	10′
9° 0′	9.1943	79	9.9946	2	9.1997	81	0.8003	**81° 0′**
10′	.2022	78	.9944	2	.2078	80	.7922	50′
20′	.2100	76	.9942	2	.2158	78	.7842	40′
30′	.2176	75	.9940	2	.2236	77	.7764	30′
40′	.2251	73	.9938	2	.2313	76	.7687	20′
50′	.2324	73	.9936	2	.2389	74	.7611	10′
10° 0′	9.2397	71	9.9934	3	9.2463	73	0.7537	**80° 0′**
10′	.2468	70	.9931	2	.2536	73	.7464	50′
20′	.2538	68	.9929	2	.2609	71	.7391	40′
30′	.2606	68	.9927	3	.2680	70	.7320	30′
40′	.2674	66	.9924	2	.2750	69	.7250	20′
50′	.2740	66	.9922	3	.2819	68	.7181	10′
11° 0′	9.2806	64	9.9919	2	9.2887	66	0.7113	**79° 0′**
10′	.2870	64	.9917	3	.2953	67	.7047	50′
20′	.2934	63	.9914	2	.3020	65	.6980	40′
30′	.2997	61	.9912	3	.3085	64	.6915	30′
40′	.3058	61	.9909	2	.3149	63	.6851	20′
50′	.3119	60	.9907	3	.3212	63	.6788	10′
12° 0′	9.3179	59	9.9904	3	9.3275	61	0.6725	**78° 0′**
10′	.3238	58	.9901	2	.3336	61	.6664	50′
20′	.3296	57	.9899	3	.3397	61	.6603	40′
30′	.3353	57	.9896	3	.3458	59	.6542	30′
40′	.3410	56	.9893	3	.3517	59	.6483	20′
50′	.3466	55	.9890	3	.3576	58	.6424	10′
13° 0′	9.3521	54	9.9887	3	9.3634	57	0.6366	**77° 0′**
10′	.3575	54	.9884	3	.3691	57	.6309	50′
20′	.3629	53	.9881	3	.3748	56	.6252	40′
30′	.3682	52	.9878	3	.3804	55	.6196	30′
40′	.3734	52	.9875	3	.3859	55	.6141	20′
50′	.3786	51	.9872	3	.3914	54	.6086	10′
14° 0′	9.3837	50	9.9869	3	9.3968	53	0.6032	**76° 0′**
10′	.3887	50	.9866	3	.4021	53	.5979	50′
20′	.3937	49	.9863	4	.4074	53	.5926	40′
30′	.3986	49	.9859	3	.4127	51	.5873	30′
40′	.4035	48	.9856	3	.4178	52	.5822	20′
50′	.4083	47	.9853	4	.4230	51	.5770	10′
15° 0′	9.4130		9.9849		9.4281		0.5719	**75° 0′**
	log cos	d	log sin	d	log cot	d	log tan	x

Prop. Parts

	73	71	70	69	68
1	7.3	7.1	7.0	6.9	6.8
2	14.6	14.2	14.0	13.8	13.6
3	21.9	21.3	21.0	20.7	20.4
4	29.2	28.4	28.0	27.6	27.2
5	36.5	35.5	35.0	34.5	34.0
6	43.8	42.6	42.0	41.4	40.8
7	51.1	49.7	49.0	48.3	47.6
8	58.4	56.8	56.0	55.2	54.4
9	65.7	63.9	63.0	62.1	61.2

	67	66	65	64	63
1	6.7	6.6	6.5	6.4	6 3
2	13.4	13.2	13.0	12.8	12.6
3	20.1	19.8	19.5	19.2	18.9
4	26.8	26.4	26.0	25.6	25.2
5	33.5	33.0	32.5	32.0	31.5
6	40.2	39.6	39.0	38.4	37.8
7	46.9	46.2	45.5	44.8	44.1
8	53.6	52.8	52.0	51.2	50.4
9	60.3	59.4	58.5	57.6	56.7

	61	60	59	58	57
1	6.1	6.0	5.9	5.8	5.7
2	12.2	12.0	11.8	11.6	11.4
3	18.3	18.0	17.7	17.4	17.1
4	24.4	24.0	23.6	23.2	22.8
5	30.5	30.0	29.5	29.0	28.5
6	36.6	36.0	35.4	34.8	34.2
7	42.7	42.0	41.3	40.6	39.9
8	48.8	48.0	47.2	46.4	45.6
9	54.9	54.0	53.1	52.2	51.3

	56	55	54	53	52
1	5.6	5.5	5.4	5.3	5.2
2	11.2	11.0	10.8	10.6	10.4
3	16.8	16.5	16.2	15.9	15.6
4	22.4	22.0	21.6	21.2	20.8
5	28.0	27.5	27.0	26.5	26.0
6	33.6	33.0	32.4	31.8	31.2
7	39.2	38.5	37.8	37.1	36.4
8	44.8	44.0	43.2	42.4	41.6
9	50.4	49.5	48.6	47.7	46.8

	51	50	48	47
1	5.1	5.0	4.8	4.7
2	10.2	10.0	9.6	9.4
3	15.3	15.0	14.4	14.1
4	20.4	20.0	19.2	18.8
5	25.5	25.0	24.0	23.5
6	30.6	30.0	28.8	28.2
7	35.7	35.0	33.6	32.9
8	40.8	40.0	38.4	37.6
9	45.9	45.5	43.2	42.3

	97	94	93	91	89	87	86	85	84	82	81	79	78	77	76	75	74
1	9.7	9.4	9.3	9.1	8.9	8.7	8.6	8.5	8.4	8.2	8.1	7.9	7.8	7.7	7.6	7.5	7.4
2	19.4	18.8	18.6	18.2	17.8	17.4	17.2	17.0	16.8	16.4	16.2	15.8	15.6	15.4	15.2	15.0	14.8
3	29.1	28.2	27.9	27.3	26.7	26.1	25.8	25.5	25.2	24.6	24.3	23.7	23.4	23.1	22.8	22.5	22.2
4	38.8	37.6	37.2	36.4	35.6	34.8	34.4	34.0	33.6	32.8	32.4	31.6	31.2	30.8	30.4	30.0	29.6
5	48.5	47.0	46.5	45.5	44.5	43.5	43.0	42.5	42.0	41.0	40.5	39.5	39.0	38.5	38.0	37.5	37.0
6	58.2	56.4	55.8	54.6	53.4	52.2	51.6	51.0	50.4	49.2	48.6	47.4	46.8	46.2	45.6	45.0	44.4
7	67.9	65.8	65.1	63.7	62.3	60.9	60.2	59.5	58.8	57.4	56.7	55.3	54.6	53.9	53.2	52.5	51.8
8	77.6	75.2	74.4	72.8	71.2	69.6	68.8	68.0	67.2	65.6	64.8	63.2	62.4	61.6	60.8	60.0	59.2
9	87.3	84.6	83.7	81.9	80.1	78.3	77.4	76.5	75.6	73.8	72.9	71.1	70.2	69.3	68.4	67.5	66.6

To be cut off

x	log sin	d	log cos	d	log tan	d	log cot	
15° 0′	9.4130	47	9.9849	3	9.4281	50	0.5719	**75° 0′**
10′	.4177	46	.9846	3	.4331	50	.5669	50′
20′	.4223	46	.9843	4	.4381	49	.5619	40′
30′	.4269	45	.9839	3	.4430	49	.5570	30′
40′	.4314	45	.9836	4	.4479	48	.5521	20′
50′	.4359	44	.9832	4	.4527	48	.5473	10′
16° 0′	9.4403	44	9.9828	3	9.4575	47	0.5425	**74° 0′**
10′	.4447	44	.9825	4	.4622	47	.5378	50′
20′	.4491	42	.9821	4	.4669	47	.5331	40′
30′	.4533	43	.9817	3	.4716	46	.5284	30′
40′	.4576	42	.9814	4	.4762	46	.5238	20′
50′	.4618	41	.9810	4	.4808	45	.5192	10′
17° 0′	9.4659	41	9.9806	4	9.4853	45	0.5147	**73° 0′**
10′	.4700	41	.9802	4	.4898	45	.5102	50′
20′	.4741	40	.9798	4	.4943	44	.5057	40′
30′	.4781	40	.9794	4	.4987	44	.5013	30′
40′	.4821	40	.9790	4	.5031	44	.4969	20′
50′	.4861	39	.9786	4	.5075	43	.4925	10′
18° 0′	9.4900	39	9.9782	4	9.5118	43	0.4882	**72° 0′**
10′	.4939	38	.9778	4	.5161	42	.4839	50′
20′	.4977	38	.9774	4	.5203	42	.4797	40′
30′	.5015	37	.9770	5	.5245	42	.4755	30′
40′	.5052	38	.9765	4	.5287	42	.4713	20′
50′	.5090	36	.9761	4	.5329	41	.4671	10′
19° 0′	9.5126	37	9.9757	5	9.5370	41	0.4630	**71° 0′**
10′	.5163	36	.9752	4	.5411	40	.4589	50′
20′	.5199	36	.9748	5	.5451	40	.4549	40′
30′	.5235	35	.9743	4	.5491	40	.4509	30′
40′	.5270	36	.9739	5	.5531	40	.4469	20′
50′	.5306	35	.9734	4	.5571	40	.4429	10′
20° 0′	9.5341	34	9.9730	5	9.5611	39	0.4389	**70° 0′**
10′	.5375	34	.9725	4	.5650	39	.4350	50′
20′	.5409	34	.9721	5	.5689	38	.4311	40′
30′	.5443	34	.9716	5	.5727	39	.4273	30′
40′	.5477	33	.9711	5	.5766	38	.4234	20′
50′	.5510	33	.9706	4	.5804	38	.4196	10′
21° 0′	9.5543	33	9.9702	5	9.5842	37	0.4158	**69° 0′**
10′	.5576	33	.9697	5	.5879	38	.4121	50′
20′	.5609	32	.9692	5	.5917	37	.4083	40′
30′	.5641	32	.9687	5	.5954	37	.4046	30′
40′	.5673	31	.9682	5	.5991	37	.4009	20′
50′	.5704	32	.9677	5	.6028	36	.3972	10′
22° 0′	9.5736	31	9.9672	5	9.6064	36	0.3936	**68° 0′**
10′	.5767	31	.9667	6	.6100	36	.3900	50′
20′	.5798	30	.9661	5	.6136	36	.3864	40′
30′	.5828		.9656		.6172		.3828	30′
	log cos	d	log sin	d	log cot	d	log tan	*x*

Prop. Parts				
	50	**49**	**48**	**47**
1	5.0	4.9	4.8	4.7
2	10.0	9.8	9.6	9.4
3	15.0	14.7	14.4	14.1
4	20.0	19.6	19.2	18.8
5	25.0	24.5	24.0	23.5
6	30.0	29.4	28.8	28.2
7	35.0	34.3	33.6	32.9
8	40.0	39.2	38.4	37.6
9	45.0	44.1	43.2	42.3
	46	**45**	**44**	**43**
1	4.6	4.5	4.4	4.3
2	9.2	9.0	8.8	8.6
3	13.8	13.5	13.2	12.9
4	18.4	18.0	17.6	17.2
5	23.0	22.5	22.0	21.5
6	27.6	27.0	26.4	25.8
7	32.2	31.5	30.8	30.1
8	36.8	36.0	35.2	34.4
9	41.4	40.5	39.6	38.7
	42	**41**	**40**	**39**
1	4.2	4.1	4.0	3.9
2	8.4	8.2	8.0	7.8
3	12.6	12.3	12.0	11.7
4	16.8	16.4	16.0	15.6
5	21.0	20.5	20.0	19.5
6	25.2	24.6	24.0	23.4
7	29.4	28.7	28.0	27.3
8	33.6	32.8	32.0	31.2
9	37.8	36.9	36.0	35.1
	38	**37**	**36**	**35**
1	3.8	3.7	3.6	3.5
2	7.6	7.4	7.2	7.0
3	11.4	11.1	10.8	10.5
4	15.2	14.8	14.4	14.0
5	19.0	18.5	18.0	17.5
6	22.8	22.2	21.6	21.0
7	26.6	25.9	25.2	24.5
8	30.4	29.6	28.8	28.0
9	34.2	33.3	32.4	31.5
	34	**33**	**32**	**31**
1	3.4	3.3	3.2	3.1
2	6.8	6.6	6.4	6.2
3	10.2	9.9	9.6	9.3
4	13.6	13.2	12.8	12.4
5	17.0	16.5	16.0	15.5
6	20.4	19.8	19.2	18.6
7	23.8	23.1	22.4	21.7
8	27.2	26.4	25.6	24.8
9	30.6	29.7	28.8	27.9
Prop. Parts				

x	log sin	d	log cos	d	log tan	d	log cot	
30′	9.5828	31	9.9656	5	9.6172	36	0.3828	30′
40′	.5859	30	.9651	5	.6208	35	.3792	20′
50′	.5889	30	.9646	6	.6243	36	.3757	10′
23° 0′	9.5919	29	9.9640	5	9.6279	35	0.3721	**67° 0′**
10′	.5948	30	.9635	6	.6314	34	.3686	50′
20′	.5978	29	.9629	5	.6348	35	.3652	40′
30′	.6007	29	.9624	6	.6383	34	.3617	30′
40′	.6036	29	.9618	5	.6417	35	.3583	20′
50′	.6065	28	.9613	6	.6452	34	.3548	10′
24° 0′	9.6093	28	9.9607	5	9.6486	34	0.3514	**66° 0′**
10′	.6121	28	.9602	6	.6520	33	.3480	50′
20′	.6149	28	.9596	6	.6553	34	.3447	40′
30′	.6177	28	.9590	6	.6587	33	.3413	30′
40′	.6205	27	.9584	5	.6620	34	.3380	20′
50′	.6232	27	.9579	6	.6654	33	.3346	10′
25° 0′	9.6259	27	9.9573	6	9.6687	33	0.3313	**65° 0′**
10′	.6286	27	.9567	6	.6720	32	.3280	50′
20′	.6313	27	.9561	6	.6752	33	.3248	40′
30′	.6340	26	.9555	6	.6785	32	.3215	30′
40′	.6366	26	.9549	6	.6817	33	.3183	20′
50′	.6392	26	.9543	6	.6850	32	.3150	10′
26° 0′	9.6418	26	9.9537	7	9.6882	32	0.3118	**64° 0′**
10′	.6444	26	.9530	6	.6914	32	.3086	50′
20′	.6470	25	.9524	6	.6946	31	.3054	40′
30′	.6495	26	.9518	6	.6977	32	.3023	30′
40′	.6521	25	.9512	7	.7009	31	.2991	20′
50′	.6546	24	.9505	6	.7040	32	.2960	10′
27° 0′	9.6570	25	9.9499	7	9.7072	31	0.2928	**63° 0′**
10′	.6595	25	.9492	6	.7103	31	.2897	50′
20′	.6620	24	.9486	7	.7134	31	.2866	40′
30′	.6644	24	.9479	6	.7165	31	.2835	30′
40′	.6668	24	.9473	7	.7196	30	.2804	20′
50′	.6692	24	.9466	7	.7226	31	.2774	10′
28° 0′	9.6716	24	9.9459	6	9.7257	30	0.2743	**62° 0′**
10′	.6740	23	.9453	7	.7287	30	.2713	50′
20′	.6763	24	.9446	7	.7317	31	.2683	40′
30′	.6787	23	.9439	7	.7348	30	.2652	30′
40′	.6810	23	.9432	7	.7378	30	.2622	20′
50′	.6833	23	.9425	7	.7408	30	.2592	10′
29° 0′	9.6856	22	9.9418	7	9.7438	29	0.2562	**61° 0′**
10′	.6878	23	.9411	7	.7467	30	.2533	50′
20′	.6901	22	.9404	7	.7497	29	.2503	40′
30′	.6923	23	.9397	7	.7526	30	.2474	30′
40′	.6946	22	.9390	7	.7556	29	.2444	20′
50′	.6968	22	.9383	8	.7585	29	.2415	10′
30° 0′	9.6990		9.9375		9.7614		0.2386	**60° 0′**
	log cos	d	log sin	d	log cot	d	log tan	x

Prop. Parts

	36	35	34
1	3.6	3.5	3.4
2	7.2	7.0	6.8
3	10.8	10.5	10.2
4	14.4	14.0	13.6
5	18.0	17.5	17.0
6	21.6	21.0	20.4
7	25.2	24.5	23.8
8	28.8	28.0	27.2
9	32.4	31.5	30.6

	33	32	31
1	3.3	3.2	3.1
2	6.6	6.4	6.2
3	9.9	9.6	9.3
4	13.2	12.8	12.4
5	16.5	16.0	15.5
6	19.8	19.2	18.6
7	23.1	22.4	21.7
8	26.4	25.6	24.8
9	29.7	28.8	27.9

	30	29	28
1	3.0	2.9	2.8
2	6.0	5.8	5.6
3	9.0	8.7	8.4
4	12.0	11.6	11.2
5	15.0	14.5	14.0
6	18.0	17.4	16.8
7	21.0	20.3	19.6
8	24.0	23.2	22.4
9	27.0	26.1	25.2

	27	26	25
1	2.7	2.6	2.5
2	5.4	5.2	5.0
3	8.1	7.8	7.5
4	10.8	10.4	10.0
5	13.5	13.0	12.5
6	16.2	15.6	15.0
7	18.9	18.2	17.5
8	21.6	20.8	20.0
9	24.3	23.4	22.5

	24	23	22
1	2.4	2.3	2.2
2	4.8	4.6	4.4
3	7.2	6.9	6.6
4	9.6	9.2	8.8
5	12.0	11.5	11.0
6	14.4	13.8	13.2
7	16.8	16.1	15.4
8	19.2	18.4	17.6
9	21.6	20.7	19.8

Prop. Parts

15° – 30°
75° – 60°

To be cut off

x	**log sin**	**d**	**log cos**	**d**	**log tan**	**d**	**log cot**	
30° 0′	9.6990	22	9.9375	7	9.7614	30	0.2386	**60° 0′**
10′	.7012	21	.9368	7	.7644	29	.2356	50′
20′	.7033	22	.9361	8	.7673	28	.2327	40′
30′	.7055	21	.9353	7	.7701	29	.2299	30′
40′	.7076	21	.9346	8	.7730	29	.2270	20′
50′	.7097	21	.9338	7	.7759	29	.2241	10′
31° 0′	9.7118	21	9.9331	8	9.7788	28	0.2212	**59° 0′**
10′	.7139	21	.9323	8	.7816	29	.2184	50′
20′	.7160	21	.9315	7	.7845	28	.2155	40′
30′	.7181	20	.9308	8	.7873	29	.2127	30′
40′	.7201	21	.9300	8	.7902	28	.2098	20′
50′	.7222	20	.9292	8	.7930	28	.2070	10′
32° 0′	9.7242	20	9.9284	8	9.7958	28	0.2042	**58° 0′**
10′	.7262	20	.9276	8	.7986	28	.2014	50′
20′	.7282	20	.9268	8	.8014	28	.1986	40′
30′	.7302	20	.9260	8	.8042	28	.1958	30′
40′	.7322	20	.9252	8	.8070	27	.1930	20′
50′	.7342	19	.9244	8	.8097	28	.1903	10′
33° 0′	9.7361	19	9.9236	8	9.8125	28	0.1875	**57° 0′**
10′	.7380	20	.9228	9	.8153	27	.1847	50′
20′	.7400	19	.9219	8	.8180	28	.1820	40′
30′	.7419	19	.9211	8	.8208	27	.1792	30′
40′	.7438	19	.9203	9	.8235	28	.1765	20′
50′	.7457	19	.9194	8	.8263	27	.1737	10′
34° 0′	9.7476	18	9.9186	9	9.8290	27	0.1710	**56° 0′**
10′	.7494	19	.9177	8	.8317	27	.1683	50′
20′	.7513	18	.9169	9	.8344	27	.1656	40′
30′	.7531	19	.9160	9	.8371	27	.1629	30′
40′	.7550	18	.9151	9	.8398	27	.1602	20′
50′	.7568	18	.9142	8	.8425	27	.1575	10′
35° 0′	9.7586	18	9.9134	9	9.8452	27	0.1548	**55° 0′**
10′	.7604	18	.9125	9	.8479	27	.1521	50′
20′	.7622	18	.9116	9	.8506	27	.1494	40′
30′	.7640	17	.9107	9	.8533	26	.1467	30′
40′	.7657	18	.9098	9	.8559	27	.1441	20′
50′	.7675	17	.9089	9	.8586	27	.1414	10′
36° 0′	9.7692	18	9.9080	10	9.8613	26	0.1387	**54° 0′**
10′	.7710	17	.9070	9	.8639	27	.1361	50′
20′	.7727	17	.9061	9	.8666	26	.1334	40′
30′	.7744	17	.9052	10	.8692	26	.1308	30′
40′	.7761	17	.9042	9	.8718	27	.1282	20′
50′	.7778	17	.9033	10	.8745	26	.1255	10′
37° 0′	9.7795	16	9.9023	9	9.8771	26	0.1229	**53° 0′**
10′	.7811	17	.9014	10	.8797	27	.1203	50′
20′	.7828	16	.9004	9	.8824	26	.1176	40′
30′	.7844		.8995		.8850		.1150	30′
	log cos	**d**	**log sin**	**d**	**log cot**	**d**	**log tan**	*x*

Prop. Parts

	30	**29**	**28**
1	3.0	2.9	2.8
2	6.0	5.8	5.6
3	9.0	8.7	8.4
4	12.0	11.6	11.2
5	15.0	14.5	14.0
6	18.0	17.4	16.8
7	21.0	20.3	19.6
8	24.0	23.2	22.4
9	27.0	26.1	25.2

	27	**26**	**22**
1	2.7	2.6	2.2
2	5.4	5.2	4.4
3	8.1	7.8	6.6
4	10.8	10.4	8.8
5	13.5	13.0	11.0
6	16.2	15.6	13.2
7	18.9	18.2	15.4
8	21.6	20.8	17.6
9	24.3	23.4	19.8

	21	**20**	**19**
1	2.1	2.0	1.9
2	4.2	4.0	3.8
3	6.3	6.0	5.7
4	8.4	8.0	7.6
5	10.5	10.0	9.5
6	12.6	12.0	11.4
7	14.7	14.0	13.3
8	16.8	16.0	15.2
9	18.9	18.0	17.1

	18	**17**	**16**
1	1.8	1.7	1.6
2	3.6	3.4	3.2
3	5.4	5.1	4.8
4	7.2	6.8	6.4
5	9.0	8.5	8.0
6	10.8	10.2	9.6
7	12.6	11.9	11.2
8	14.4	13.6	12.8
9	16.2	15.3	14.4

	9	**8**	**7**
1	.9	.8	.7
2	1.8	1.6	1.4
3	2.7	2.4	2.1
4	3.6	3.2	2.8
5	4.5	4.0	3.5
6	5.4	4.8	4.2
7	6.3	5.6	4.9
8	7.2	6.4	5.6
9	8.1	7.2	6.3

Prop. Parts

x	**log sin**	**d**	**log cos**	**d**	**log tan**	**d**	**log cot**	
30′	9.7844	17	9.8995	10	9.8850	26	0.1150	30′
40′	.7861	16	.8985	10	.8876	26	.1124	20′
50′	.7877	16	.8975	10	.8902	26	.1098	10′
38° 0′	9.7893	17	9.8965	10	9.8928	26	0.1072	**52° 0′**
10′	.7910	16	.8955	10	.8954	26	.1046	50′
20′	.7926	15	.8945	10	.8980	26	.1020	40′
30′	.7941	16	.8935	10	.9006	26	.0994	30′
40′	.7957	16	.8925	10	.9032	26	.0968	20′
50′	.7973	16	.8915	10	.9058	26	.0942	10′
39° 0′	9.7989	15	9.8905	10	9.9084	26	0.0916	**51° 0′**
10′	.8004	16	.8895	11	.9110	25	.0890	50′
20′	.8020	15	.8884	10	.9135	26	.0865	40′
30′	.8035	15	.8874	10	.9161	26	.0839	30′
40′	.8050	16	.8864	11	.9187	25	.0813	20′
50′	.8066	15	.8853	10	.9212	26	.0788	10′
40° 0′	9.8081	15	9.8843	11	9.9238	26	0.0762	**50° 0′**
10′	.8096	15	.8832	11	.9264	25	.0736	50′
20′	.8111	14	.8821	11	.9289	26	.0711	40′
30′	.8125	15	.8810	10	.9315	26	.0685	30′
40′	.8140	15	.8800	11	.9341	25	.0659	20′
50′	.8155	14	.8789	11	.9366	26	.0634	10′
41° 0′	9.8169	15	9.8778	11	9.9392	25	0.0608	**49° 0′**
10′	.8184	14	.8767	11	.9417	26	.0583	50′
20′	.8198	15	.8756	11	.9443	25	.0557	40′
30′	.8213	14	.8745	12	.9468	26	.0532	30′
40′	.8227	14	.8733	11	.9494	25	.0506	20′
50′	.8241	14	.8722	11	.9519	25	.0481	10′
42° 0′	9.8255	14	9.8711	12	9.9544	26	0.0456	**48° 0′**
10′	.8269	14	.8699	11	.9570	25	.0430	50′
20′	.8283	14	.8688	12	.9595	26	.0405	40′
30′	.8297	14	.8676	11	.9621	25	.0379	30′
40′	.8311	13	.8665	12	.9646	25	.0354	20′
50′	.8324	14	.8653	12	.9671	26	.0329	10′
43° 0′	9.8338	13	9.8641	12	9.9697	25	0.0303	**47° 0′**
10′	.8351	14	.8629	11	.9722	25	.0278	50′
20′	.8365	13	.8618	12	.9747	25	.0253	40′
30′	.8378	13	.8606	12	.9772	26	.0228	30′
40′	.8391	14	.8594	12	.9798	25	.0202	20′
50′	.8405	13	.8582	13	.9823	25	.0177	10′
44° 0′	9.8418	13	9.8569	12	9.9848	26	0.0152	**46° 0′**
10′	.8431	13	.8557	12	.9874	25	.0126	50′
20′	.8444	13	.8545	13	.9899	25	.0101	40′
30′	.8457	12	.8532	12	.9924	25	.0076	30′
40′	.8469	13	.8520	13	.9949	26	.0051	20′
50′	.8482	13	.8507	12	.9975	25	.0025	10′
45° 0′	9.8495		9.8495		0.0000		0.0000	**45° 0′**
	log cos	**d**	**log sin**	**d**	**log cot**	**d**	**log tan**	*x*

Prop. Parts

	26	**25**
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	17	**16**	**15**
1	1.7	1.6	1.5
2	3.4	3.2	3.0
3	5.1	4.8	4.5
4	6.8	6.4	6.0
5	8.5	8.0	7.5
6	10.2	9.6	9.0
7	11.9	11.2	10.5
8	13.6	12.8	12.0
9	15.3	14.4	13.5

	14	**13**	**12**
1	1.4	1.3	1.2
2	2.8	2.6	2.4
3	4.2	3.9	3.6
4	5.6	5.2	4.8
5	7.0	6.5	6.0
6	8.4	7.8	7.2
7	9.8	9.1	8.4
8	11.2	10.4	9.6
9	12.6	11.7	10.8

	11	**10**
1	1.1	1.0
2	2.2	2.0
3	3.3	3.0
4	4.4	4.0
5	5.5	5.0
6	6.6	6.0
7	7.7	7.0
8	8.8	8.0
9	9.9	9.0

Prop. Parts

To be cut off

x	sin *x*	cos *x*	tan *x*	cot *x*	sec *x*	cosec *x*	
0° 0′	.00000	1.0000	.00000	∞	1.0000	∞	90° 0′
10′	.00291	1.0000	.00291	343.77	1.0000	343.78	50′
20′	.00582	1.0000	.00582	171.88	1.0000	171.89	40′
30′	.00873	1.0000	.00873	114.59	1.0000	114.59	30′
40′	.01164	.9999	.01164	85.940	1.0001	85.946	20′
50′	.01454	.9999	.01455	68.750	1.0001	68.757	10′
1° 0′	.01745	.9998	.01746	57.290	1.0002	57.299	89° 0′
10′	.02036	.9998	.02036	49.104	1.0002	49.114	50′
20′	.02327	.9997	.02328	42.964	1.0003	42.976	40′
30′	.02618	.9997	.02619	38.188	1.0003	38.202	30′
40′	.02908	.9996	.02910	34.368	1.0004	34.382	20′
50′	.03199	.9995	.03201	31.242	1.0005	31.258	10′
2° 0′	.03490	.9994	.03492	28.6363	1.0006	28.654	88° 0′
10′	.03781	.9993	.03783	26.4316	1.0007	26.451	50′
20′	.04071	.9992	.04075	24.5418	1.0008	24.562	40′
30′	.04362	.9990	.04366	22.9038	1.0010	22.926	30′
40′	.04653	.9989	.04658	21.4704	1.0011	21.494	20′
50′	.04943	.9988	.04949	20.2056	1.0012	20.230	10′
3° 0′	.05234	.9986	.05241	19.0811	1.0014	19.107	87° 0′
10′	.05524	.9985	.05533	18.0750	1.0015	18.103	50′
20′	.05814	.9983	.05824	17.1693	1.0017	17.198	40′
30′	.06105	.9981	.06116	16.3499	1.0019	16.380	30′
40′	.06395	.9980	.06408	15.6048	1.0021	15.637	20′
50′	.06685	.9978	.06700	14.9244	1.0022	14.958	10′
4° 0′	.06976	.9976	.06993	14.3007	1.0024	14.336	86° 0′
10′	.07266	.9974	.07285	13.7267	1.0027	13.763	50′
20′	.07556	.9971	.07578	13.1969	1.0029	13.235	40′
30′	.07846	.9969	.07870	12.7062	1.0031	12.746	30′
40′	.08136	.9967	.08163	12.2505	1.0033	12.291	20′
50′	.08426	.9964	.08456	11.8262	1.0036	11.868	10′
5° 0′	.08716	.9962	.08749	11.4301	1.0038	11.474	85° 0′
10′	.09005	.9959	.09042	11.0594	1.0041	11.105	50′
20′	.09295	.9957	.09335	10.7119	1.0044	10.758	40′
30′	.09585	.9954	.09629	10.3854	1.0046	10.433	30′
40′	.09874	.9951	.09923	10.0780	1.0049	10.128	20′
50′	.10164	.9948	.10216	9.7882	1.0052	9.839	10′
6° 0′	.10453	.9945	.10510	9.5144	1.0055	9.5668	84° 0′
10′	.10742	.9942	.10805	9.2553	1.0058	9.3092	50′
20′	.11031	.9939	.11099	9.0098	1.0061	9.0652	40′
30′	.11320	.9936	.11394	8.7769	1.0065	8.8337	30′
40′	.11609	.9932	.11688	8.5555	1.0068	8.6138	20′
50′	.11898	.9929	.11983	8.3450	1.0072	8.4647	10′
7° 0′	.12187	.9925	.12278	8.1443	1.0075	8.2055	83° 0′
10′	.12476	.9922	.12574	7.9530	1.0079	8.0157	50′
20′	.12764	.9918	.12869	7.7704	1.0083	7.8344	40′
30′	.13053	.9914	.13165	7.5958	1.0086	7.6613	30′
	cos *x*	sin *x*	cot *x*	tan *x*	cosec *x*	sec *x*	*x*

x	sin *x*	cos *x*	tan *x*	cot *x*	sec *x*	cosec *x*	
30′	.1305	.9914	.1317	7.5958	1.0086	7.6613	30′
40′	.1334	.9911	.1346	7.4287	1.0090	7.4957	20′
50′	.1363	.9907	.1376	7.2687	1.0094	7.3372	10′
8° 0′	.1392	.9903	.1405	7.1154	1.0098	7.1853	**82°** 0′
10′	.1421	.9899	.1435	6.9682	1.0102	7.0396	50′
20′	.1449	.9894	.1465	6.8269	1.0107	6.8998	40′
30′	.1478	.9890	.1495	6.6912	1.0111	6.7655	30′
40′	.1507	.9886	.1524	6.5606	1.0116	6.6363	20′
50′	.1536	.9881	.1554	6.4348	1.0120	6.5121	10′
9° 0′	.1564	.9877	.1584	6.3138	1.0125	6.3925	**81°** 0′
10′	.1593	.9872	.1614	6.1970	1.0129	6.2772	50′
20′	.1622	.9868	.1644	6.0844	1.0134	6.1661	40′
30′	.1650	.9863	.1673	5.9758	1.0139	6.0589	30′
40′	.1679	.9858	.1703	5.8708	1.0144	5.9554	20′
50′	.1708	.9853	.1733	5.7694	1.0149	5.8554	10′
10° 0′	.1736	.9848	.1763	5.6713	1.0154	5.7588	**80°** 0′
10′	.1765	.9843	.1793	5.5764	1.0160	5.6653	50′
20′	.1794	.9838	.1823	5.4845	1.0165	5.5749	40′
30′	.1822	.9833	.1853	5.3955	1.0170	5.4874	30′
40′	.1851	.9827	.1883	5.3093	1.0176	5.4026	20′
50′	.1880	.9822	.1914	5.2257	1.0182	5.3205	10′
11° 0′	.1908	.9816	.1944	5.1446	1.0187	5.2408	**79°** 0′
10′	.1937	.9811	.1974	5.0658	1.0193	5.1636	50′
20′	.1965	.9805	.2004	4.9894	1.0199	5.0886	40′
30′	.1994	.9799	.2035	4.9152	1.0205	5.0159	30′
40′	.2022	.9793	.2065	4.8430	1.0211	4.9452	20′
50′	.2051	.9787	.2095	4.7729	1.0217	4.8765	10′
12° 0′	.2079	.9781	.2126	4.7046	1.0223	4.8097	**78°** 0′
10′	.2108	.9775	.2156	4.6382	1.0230	4.7448	50′
20′	.2136	.9769	.2186	4.5736	1.0236	4.6817	40′
30′	.2164	.9763	.2217	4.5107	1.0243	4.6202	30′
40′	.2193	.9757	.2247	4.4494	1.0249	4.5604	20′
50′	.2221	.9750	.2278	4.3897	1.0256	4.5022	10′
13° 0′	.2250	.9744	.2309	4.3315	1.0263	4.4454	**77°** 0′
10′	.2278	.9737	.2339	4.2747	1.0270	4.3901	50′
20′	.2306	.9730	.2370	4.2193	1.0277	4.3362	40′
30′	.2334	.9724	.2401	4.1653	1.0284	4.2837	30′
40′	.2363	.9717	.2432	4.1126	1.0291	4.2324	20′
50′	.2391	.9710	.2462	4.0611	1.0299	4.1824	10′
14° 0′	.2419	.9703	.2493	4.0108	1.0306	4.1336	**76°** 0′
10′	.2447	.9696	.2524	3.9617	1.0314	4.0859	50′
20′	.2476	.9689	.2555	3.9136	1.0321	4.0394	40′
30′	.2504	.9681	.2586	3.8667	1.0329	3.9939	30′
40′	.2532	.9674	.2617	3.8208	1.0337	3.9495	20′
50′	.2560	.9667	.2648	3.7760	1.0345	3.9061	10′
15° 0′	.2588	.9659	.2679	3.7321	1.0353	3.8637	**75°** 0′
	cos *x*	sin *x*	cot *x*	tan *x*	cosec *x*	sec *x*	*x*

Nat. trig. funct's.
0° – 15°
90° – 75°

To be cut off

x	**sin *x***	**cos *x***	**tan *x***	**cot *x***	**sec *x***	**cosec *x***	
15° 0′	.2588	.9659	.2679	3.7321	1.0353	3.8637	**75° 0′**
10′	.2616	.9652	.2711	3.6891	1.0361	3.8222	50′
20′	.2644	.9644	.2742	3.6470	1.0369	3.7817	40′
30′	.2672	.9636	.2773	3.6059	1.0377	3.7420	30′
40′	.2700	.9628	.2805	3.5656	1.0386	3.7032	20′
50′	.2728	.9621	.2836	3.5261	1.0394	3.6652	10′
16° 0′	.2756	.9613	.2867	3.4874	1.0403	3.6280	**74° 0′**
10′	.2784	.9605	.2899	3.4495	1.0412	3.5915	50′
20′	.2812	.9596	.2931	3.4124	1.0421	3.5559	40′
30′	.2840	.9588	.2962	3.3759	1.0430	3.5209	30′
40′	.2868	.9580	.2994	3.3402	1.0439	3.4867	20′
50′	.2896	.9572	.3026	3.3052	1.0448	3.4532	10′
17° 0′	.2924	.9563	.3057	3.2709	1.0457	3.4203	**73° 0′**
10′	.2952	.9555	.3089	3.2371	1.0466	3.3881	50′
20′	.2979	.9546	.3121	3.2041	1.0476	3.3565	40′
30′	.3007	.9537	.3153	3.1716	1.0485	3.3255	30′
40′	.3035	.9528	.3185	3.1397	1.0495	3.2951	20′
50′	.3062	.9520	.3217	3.1084	1.0505	3.2653	10′
18° 0′	.3090	.9511	.3249	3.0777	1.0515	3.2361	**72° 0′**
10′	.3118	.9502	.3281	3.0475	1.0525	3.2074	50′
20′	.3145	.9492	.3314	3.0178	1.0535	3.1792	40′
30′	.3173	.9483	.3346	2.9887	1.0545	3.1516	30′
40′	.3201	.9474	.3378	2.9600	1.0555	3.1244	20′
50′	.3228	.9465	.3411	2.9319	1.0566	3.0977	10′
19° 0′	.3256	.9455	.3443	2.9042	1.0576	3.0716	**71° 0′**
10′	.3283	.9446	.3476	2.8770	1.0587	3.0458	50′
20′	.3311	.9436	.3508	2.8502	1.0598	3.0206	40′
30′	.3338	.9426	.3541	2.8239	1.0609	2.9957	30′
40′	.3365	.9417	.3574	2.7980	1.0620	2.9714	20′
50′	.3393	.9407	.3607	2.7725	1.0631	2.9474	10′
20° 0′	.3420	.9397	.3640	2.7475	1.0642	2.9238	**70° 0′**
10′	.3448	.9387	.3673	2.7228	1.0653	2.9006	50′
20′	.3475	.9377	.3706	2.6985	1.0665	2.8779	40′
30′	.3502	.9367	.3739	2.6746	1.0676	2.8555	30′
40′	.3529	.9356	.3772	2.6511	1.0688	2.8334	20′
50′	.3557	.9346	.3805	2.6279	1.0700	2.8118	10′
21° 0′	.3584	.9336	.3839	2.6051	1.0712	2.7904	**69° 0′**
10′	.3611	.9325	.3872	2.5826	1.0724	2.7695	50′
20′	.3638	.9315	.3906	2.5605	1.0736	2.7488	40′
30′	.3665	.9304	.3939	2.5386	1.0748	2.7285	30′
40′	.3692	.9293	.3973	2.5172	1.0760	2.7085	20′
50′	.3719	.9283	.4006	2.4960	1.0773	2.6888	10′
22° 0′	.3746	.9272	.4040	2.4751	1.0785	2.6695	**68° 0′**
10′	.3773	.9261	.4074	2.4545	1.0798	2.6504	50′
20′	.3800	.9250	.4108	2.4342	1.0811	2.6316	40′
30′	.3827	.9239	.4142	2.4142	1.0824	2.6131	30′
	cos *x*	**sin *x***	**cot *x***	**tan *x***	**cosec *x***	**sec *x***	*x*

x	sin *x*	cos *x*	tan *x*	cot *x*	sec *x*	cosec *x*	
30′	.3827	.9239	.4142	2.4142	1.0824	2.6131	30′
40′	.3854	.9228	.4176	2.3945	1.0837	2.5949	20′
50′	.3881	.9216	.4210	2.3750	1.0850	2.5770	10′
23° 0′	.3907	.9205	.4245	2.3559	1.0864	2.5593	**67° 0′**
10′	.3934	.9194	.4279	2.3369	1.0877	2.5419	50′
20′	.3961	.9182	.4314	2.3183	1.0891	2.5247	40′
30′	.3987	.9171	.4348	2.2998	1.0904	2.5078	30′
40′	.4014	.9159	.4383	2.2817	1.0918	2.4912	20′
50′	.4041	.9147	.4417	2.2637	1.0932	2.4748	10′
24° 0′	.4067	.9135	.4452	2.2460	1.0946	2.4586	**66° 0′**
10′	.4094	.9124	.4487	2.2286	1.0961	2.4426	50′
20′	.4120	.9112	.4522	2.2113	1.0975	2.4269	40′
30′	.4147	.9100	.4557	2.1943	1.0990	2.4114	30′
40′	.4173	.9088	.4592	2.1775	1.1004	2.3961	20′
50′	.4200	.9075	.4628	2.1609	1.1019	2.3811	10′
25° 0′	.4226	.9063	.4663	2.1445	1.1034	2.3662	**65° 0′**
10′	.4253	.9051	.4699	2.1283	1.1049	2.3515	50′
20′	.4279	.9038	.4734	2.1123	1.1064	2.3371	40′
30′	.4305	.9026	.4770	2.0965	1.1079	2.3228	30′
40′	.4331	.9013	.4806	2.0809	1.1095	2.3088	20′
50′	.4358	.9001	.4841	2.0655	1.1110	2.2949	10′
26° 0′	.4384	.8988	.4877	2.0503	1.1126	2.2812	**64° 0′**
10′	.4410	.8975	.4913	2.0353	1.1142	2.2677	50′
20′	.4436	.8962	.4950	2.0204	1.1158	2.2543	40′
30′	.4462	.8949	.4986	2.0057	1.1174	2.2412	30′
40′	.4488	.8936	.5022	1.9912	1.1190	2.2282	20′
50′	.4514	.8923	.5059	1.9768	1.1207	2.2154	10′
27° 0′	.4540	.8910	.5095	1.9626	1.1223	2.2027	**63° 0′**
10′	.4566	.8897	.5132	1.9486	1.1240	2.1902	50′
20′	.4592	.8884	.5169	1.9347	1.1257	2.1779	40′
30′	.4617	.8870	.5206	1.9210	1.1274	2.1657	30′
40′	.4643	.8857	.5243	1.9074	1.1291	2.1537	20′
50′	.4669	.8843	.5280	1.8940	1.1308	2.1418	10′
28° 0′	.4695	.8829	.5317	1.8807	1.1326	2.1301	**62° 0′**
10′	.4720	.8816	.5354	1.8676	1.1343	2.1185	50′
20′	.4746	.8802	.5392	1.8546	1.1361	2.1070	40′
30′	.4772	.8788	.5430	1.8418	1.1379	2.0957	30′
40′	.4797	.8774	.5467	1.8291	1.1397	2.0846	20′
50′	.4823	.8760	.5505	1.8165	1.1415	2.0736	10′
29° 0′	.4848	.8746	.5543	1.8040	1.1434	2.0627	**61° 0′**
10′	.4874	.8732	.5581	1.7917	1.1452	2.0519	50′
20′	.4899	.8718	.5619	1.7796	1.1471	2.0413	40′
30′	.4924	.8704	.5658	1.7675	1.1490	2.0308	30′
40′	.4950	.8689	.5696	1.7556	1.1509	2.0204	20′
50′	.4975	.8675	.5735	1.7437	1.1528	2.0101	10′
30° 0′	.5000	.8660	.5774	1.7321	1.1547	2.0000	**60° 0′**
	cos *x*	sin *x*	cot *x*	tan *x*	cosec *x*	sec *x*	*x*

15° – 30°
75° – 60°

To be cut off

x	sin *x*	cos *x*	tan *x*	cot *x*	sec *x*	cosec *x*	
30° 0′	.5000	.8660	.5774	1.7321	1.1547	2.0000	**60°** 0′
10′	.5025	.8646	.5812	1.7205	1.1567	1.9900	50′
20′	.5050	.8631	.5851	1.7090	1.1586	1.9801	40′
30′	.5075	.8616	.5890	1.6977	1.1606	1.9703	30′
40′	.5100	.8601	.5930	1.6864	1.1626	1.9606	20′
50′	.5125	.8587	.5969	1.6753	1.1646	1.9511	10′
31° 0′	.5150	.8572	.6009	1.6643	1.1666	1.9416	**59°** 0′
10′	.5175	.8557	.6048	1.6534	1.1687	1.9323	50′
20′	.5200	.8542	.6088	1.6426	1.1708	1.9230	40′
30′	.5225	.8526	.6128	1.6319	1.1728	1.9139	30′
40′	.5250	.8511	.6168	1.6212	1.1749	1.9049	20′
50′	.5275	.8496	.6208	1.6107	1.1770	1.8959	10′
32° 0′	.5299	.8480	.6249	1.6003	1.1792	1.8871	**58°** 0′
10′	.5324	.8465	.6289	1.5900	1.1813	1.8783	50′
20′	.5348	.8450	.6330	1.5798	1.1835	1.8699	40′
30′	.5373	.8434	.6371	1.5697	1.1857	1.8612	30′
40′	.5398	.8418	.6412	1.5597	1.1879	1.8527	20′
50′	.5422	.8403	.6453	1.5497	1.1901	1.8444	10′
33° 0′	.5446	.8387	.6494	1.5399	1.1924	1.8361	**57°** 0′
10′	.5471	.8371	.6536	1.5301	1.1946	1.8279	50′
20′	.5495	.8355	.6577	1.5204	1.1969	1.8198	40′
30′	.5519	.8339	.6619	1.5108	1.1992	1.8118	30′
40′	.5544	.8323	.6661	1.5013	1.2015	1.8039	20′
50′	.5568	.8307	.6703	1.4919	1.2039	1.7960	10′
34° 0′	.5592	.8290	.6745	1.4826	1.2062	1.7883	**56°** 0′
10′	.5616	.8274	.6787	1.4733	1.2086	1.7806	50′
20′	.5640	.8258	.6830	1.4641	1.2110	1.7730	40′
30′	.5664	.8241	.6873	1.4550	1.2134	1.7655	30′
40′	.5688	.8225	.6916	1.4460	1.2158	1.7581	20′
50′	.5712	.8208	.6959	1.4370	1.2183	1.7507	10′
35° 0′	.5736	.8192	.7002	1.4281	1.2208	1.7435	**55°** 0′
10′	.5760	.8175	.7046	1.4193	1.2233	1.7362	50′
20′	.5783	.8158	.7089	1.4106	1.2258	1.7291	40′
30′	.5807	.8141	.7133	1.4019	1.2283	1.7221	30′
40′	.5831	.8124	.7177	1.3934	1.2309	1.7151	20′
50′	.5854	.8107	.7221	1.3848	1.2335	1.7082	10′
36° 0′	.5878	.8090	.7265	1.3764	1.2361	1.7013	**54°** 0′
10′	.5901	.8073	.7310	1.3680	1.2387	1.6945	50′
20′	.5925	.8056	.7355	1.3597	1.2413	1.6878	40′
30′	.5948	.8039	.7400	1.3514	1.2440	1.6812	30′
40′	.5972	.8021	.7445	1.3432	1.2467	1.6746	20′
50′	.5995	.8004	.7490	1.3351	1.2494	1.6681	10′
37° 0′	.6018	.7986	.7536	1.3270	1.2521	1.6616	**53°** 0′
10′	.6041	.7969	.7581	1.3190	1.2549	1.6553	50′
20′	.6065	.7951	.7627	1.3111	1.2577	1.6489	40′
30′	.6088	.7934	.7673	1.3032	1.2605	1.6427	30′
	cos *x*	sin *x*	cot *x*	tan *x*	cosec *x*	sec *x*	*x*

x	sin *x*	cos *x*	tan *x*	cot *x*	sec *x*	cosec *x*	
30′	.6088	.7934	.7673	1.3032	1.2605	1.6427	30′
40′	.6111	.7916	.7720	1.2954	1.2633	1.6365	20′
50′	.6134	.7898	.7766	1.2876	1.2662	1.6304	10′
38° 0′	.6157	.7880	.7813	1.2799	1.2690	1.6243	**52° 0′**
10′	.6180	.7862	.7860	1.2723	1.2719	1.6183	50′
20′	.6202	.7844	.7907	1.2647	1.2748	1.6123	40′
30′	.6225	.7826	.7954	1.2572	1.2779	1.6064	30′
40′	.6248	.7808	.8002	1.2497	1.2808	1.6005	20′
50′	.6271	.7790	.8050	1.2423	1.2837	1.5948	10′
39° 0′	.6293	.7771	.8098	1.2349	1.2868	1.5890	**51° 0′**
10′	.6316	.7753	.8146	1.2276	1.2898	1.5833	50′
20′	.6338	.7735	.8195	1.2203	1.2929	1.5777	40′
30′	.6361	.7716	.8243	1.2131	1.2960	1.5721	30′
40′	.6383	.7698	.8292	1.2059	1.2991	1.5666	20′
50′	.6406	.7679	.8342	1.1988	1.3022	1.5611	10′
40° 0′	.6428	.7660	.8391	1.1918	1.3054	1.5557	**50° 0′**
10′	.6450	.7642	.8441	1.1847	1.3086	1.5504	50′
20′	.6472	.7623	.8491	1.1778	1.3118	1.5450	40′
30′	.6494	.7604	.8541	1.1708	1.3151	1.5398	30′
40′	.6517	.7585	.8591	1.1640	1.3184	1.5346	20′
50′	.6539	.7566	.8642	1.1571	1.3217	1.5294	10′
41° 0′	.6561	.7547	.8693	1.1504	1.3250	1.5243	**49° 0′**
10′	.6583	.7528	.8744	1.1436	1.3284	1.5192	50′
20′	.6604	.7509	.8796	1.1369	1.3318	1.5142	40′
30′	.6626	.7490	.8847	1.1303	1.3352	1.5092	30′
40′	.6648	.7470	.8899	1.1237	1.3386	1.5042	20′
50′	.6670	.7451	.8952	1.1171	1.3421	1.4993	10′
42° 0′	.6691	.7431	.9004	1.1106	1.3456	1.4945	**48° 0′**
10′	.6713	.7412	.9057	1.1041	1.3492	1.4897	50′
20′	.6734	.7392	.9110	1.0977	1.3527	1.4849	40′
30′	.6756	.7373	.9163	1.0913	1.3563	1.4802	30′
40′	.6777	.7353	.9217	1.0850	1.3600	1.4755	20′
50′	.6799	.7333	.9271	1.0786	1.3636	1.4709	10′
43° 0′	.6820	.7314	.9325	1.0724	1.3673	1.4663	**47° 0′**
10′	.6841	.7294	.9380	1.0661	1.3711	1.4617	50′
20′	.6862	.7274	.9435	1.0599	1.3748	1.4572	40′
30′	.6884	.7254	.9490	1.0538	1.3786	1.4527	30′
40′	.6905	.7234	.9545	1.0477	1.3824	1.4483	20′
50′	.6926	.7214	.9601	1.0416	1.3863	1.4439	10′
44° 0′	.6947	.7193	.9657	1.0355	1.3902	1.4396	**46° 0′**
10′	.6967	.7173	.9713	1.0295	1.3941	1.4352	50′
20′	.6988	.7153	.9770	1.0235	1.3980	1.4310	40′
30′	.7009	.7133	.9827	1.0176	1.4020	1.4267	30′
40′	.7030	.7112	.9884	1.0117	1.4061	1.4225	20′
50′	.7050	.7092	.9942	1.0058	1.4101	1.4184	10′
45° 0′	.7071	.7071	1.0000	1.0000	1.4142	1.4142	**45° 0′**
	cos *x*	sin *x*	cot *x*	tan *x*	cosec *x*	sec *x*	*x*

30° – 45°
60° – 45°

To be cut off

TABLE IV. DEGREES TO RADIANS AND CONVERSELY

n	*n* degrees into radians	*n* minutes into radians	*n* seconds into radians	*n*	*n* radians into degree measure
0	0.00000	0.00000	0.00000		
1	0.01745	0.00029	0.00000	0.00001	0° 0′ 02″
2	0.03491	0.00058	0.00001	0.00002	0 0 04
3	0.05236	0.00087	0.00001	0.00003	0 0 06
4	0.06981	0.00116	0.00002	0.00004	0 0 08
5	0.08727	0.00145	0.00002	0.00005	0° 0′ 10″
6	0.10472	0.00175	0.00003	0.00006	0 0 12
7	0.12217	0.00204	0.00003	0.00007	0 0 14
8	0.13963	0.00233	0.00004	0.00008	0 0 17
9	0.15708	0.00262	0.00004	0.00009	0 0 19
10	0.17453	0.00291	0.00005		
11	0.19199	0.00320	0.00005	0.0001	0° 0′ 21″
12	0.20944	0.00349	0.00006	0.0002	0 0 41″
13	0.22689	0.00378	0.00006	0.0003	0 1 02
14	0.24435	0.00407	0.00007	0.0004	0 1 23
15	0.26180	0.00436	0.00007	0.0005	0° 1′ 43″
16	0.27925	0.00465	0.00008	0.0006	0 2 04
17	0.29671	0.00495	0.00008	0.0007	0 2 24
18	0.31416	0.00524	0.00009	0.0008	0 2 45
19	0.33161	0.00553	0.00009	0.0009	0 3 06
20	0.34907	0.00582	0.00010		
21	0.36652	0.00611	0.00010	0.001	0° 03′ 26″
22	0.38397	0.00640	0.00011	0.002	0 06 53
23	0.40143	0.00669	0.00011	0.003	0 10 19
24	0.41888	0.00698	0.00012	0.004	0 13 45
25	0.43633	0.00727	0.00012	0.005	0° 17′ 11″
26	0.45379	0.00756	0.00013	0.006	0 20 38
27	0.47124	0.00785	0.00013	0.007	0 24 04
28	0.48869	0.00814	0.00014	0.008	0 27 30
29	0.50615	0.00844	0.00014	0.009	0 30 56
30	0.52360	0.00873	0.00015		
31	0.54105	0.00902	0.00015	0.01	0° 34′ 23″
32	0.55851	0.00931	0.00016	0.02	1 08 45
33	0.57596	0.00960	0.00016	0.03	1 43 08
34	0.59341	0.00989	0.00016	0.04	2 17 31
35	0.61087	0.01018	0.00017	0.05	2° 51′ 53″
36	0.62832	0.01047	0.00017	0.06	3 26 16
37	0.64577	0.01076	0.00018	0.07	4 00 39
38	0.66323	0.01105	0.00018	0.08	4 35 01
39	0.68068	0.01134	0.00019	0.09	5 09 24
40	0.69813	0.01164	0.00019		
41	0.71558	0.01193	0.00020	0.1	5° 43 46″
42	0.73304	0.01222	0.00020	0.2	11 27 33
43	0.75049	0.01251	0.00021	0.3	17 11 19
44	0.76794	0.01280	0.00021	0.4	22 55 6

n	n degrees into radians	n minutes into radians	n seconds into radians	n	n radians into degree measure
45	0.78540	0.01309	0.00022	0.5	28° 38′ 52″
46	0.80285	0.01338	0.00022	0.6	34 22 39
47	0.82030	0.01367	0.00023	0.7	40 06 25
48	0.83776	0.01396	0.00023	0.8	45 50 12
49	0.85521	0.01425	0.00024	0.9	51 33 58
50	0.87266	0.01454	0.00024		
51	0.89012	0.01484	0.00025	1.0	57° 17′ 45″
52	0.90757	0.01513	0.00025	2.0	114 35 30
53	0.92502	0.01542	0.00026	3.0	171 53 14
54	0.94248	0.01571	0.00026	4.0	229 10 59
55	0.95993	0.01600	0.00027	5.0	286° 28′ 44″
56	0.97738	0.01629	0.00027	6.0	343 46 29
57	0.99484	0.01658	0.00028	7.0	401 04 14
58	1.01229	0.01687	0.00028	8.0	458 21 58
59	1.02974	0.01716	0.00029	9.0	515 39 43
60	1.04720	0.01745	0.00029	10.0	572° 57′ 28″

TABLE V. MATHEMATICAL CONSTANTS

$$\pi = 3.14159\ 26535\ 89793. \qquad \frac{1}{\pi} = 0.31830\ 98861\ 83791.$$

$$\pi^2 = 9.86960\ 44010\ 89359. \qquad \frac{1}{\pi^2} = 0.10132\ 11836\ 42338.$$

$$\pi^3 = 31.00627\ 66802\ 99820. \qquad \frac{1}{\pi^3} = 0.03225\ 15344\ 33199.$$

$$\sqrt{\pi} = 1.77245\ 38509\ 05516. \qquad \frac{1}{\sqrt{\pi}} = 0.56418\ 95835\ 47756.$$

$$1 \text{ radian} = \frac{180^\circ}{\pi} = 57^\circ.29577\ 95131,$$

$$= \frac{10800'}{\pi} = 3437'.74677\ 07849,$$

$$= \frac{648000''}{\pi} = 206264''.80624\ 70964.$$

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radians.	radians.
$1^\circ = 0.01745\ 32925\ 19943.$	$1' = 0.00029\ 08882\ 08666.$
$(1^\circ)^2 = 0.00030\ 46174\ 19787.$	$(1')^2 = 0.00000\ 00846\ 15950.$
$(1^\circ)^3 = 0.00000\ 53165\ 76934.$	$(1')^3 = 0.00000\ 00000\ 24614.$

$$1'' = 0.00000\ 48481\ 36811.$$

$$(1'')^2 = 0.00000\ 00000\ 23504.$$

$$\sin 1^\circ = 0.01745\ 24064\ 37284.$$

$$\sin 1' = 0.00029\ 08882\ 04563.$$

$$\sin 1'' = 0.00000\ 48481\ 36811.$$

$$e = \text{Naperian base} = 1 + \frac{1}{\lfloor\underline{2}} + \frac{1}{\lfloor\underline{3}} + \ldots = 2.71828\ 18284\ 59045.$$

$$M = 0.43429\ 44819\ 03252; \quad \log_{10} n = M \log_e n.$$

$$\frac{1}{M} = 2.30258\ 50929\ 94046; \quad \log_e n = \frac{1}{M}\log_{10} n.$$

Radians to degrees and conversely. Math. const's.

To be cut off

TABLE VI. VALUES OF LOG$_e$ x, e^x AND e^{-x}.

x	$\log_e x$	e^x	e^{-x}	x	$\log_e x$	e^x	e^{-x}
0.00	$-\infty$	1.000	1.000	2.50	0.916	12.18	0.082
0.05	−2.996	1.051	0.951	2.55	0.936	12.81	0.078
0.10	−2.303	1.105	0.905	2.60	0.956	13.46	0.074
0.15	−1.897	1.162	0.861	2.65	0.975	14.15	0.071
0.20	−1.610	1.221	0.819	2.70	0.993	14.88	0.067
0.25	−1.386	1.284	0.779	2.75	1.012	15.64	0.064
0.30	−1.204	1.350	0.741	2.80	1.030	16.44	0.061
0.35	−1.050	1.419	0.705	2.85	1.047	17.29	0.058
0.40	−0.916	1.492	0.670	2.90	1.065	18.17	0.055
0.45	−0.799	1.568	0.638	2.95	1.082	19.11	0.052
0.50	−0.693	1.649	0.607	3.00	1.099	20.09	0.050
0.55	−0.598	1.733	0.577	3.05	1.115	21.12	0.047
0.60	−0.511	1.822	0.549	3.10	1.131	22.20	0.045
0.65	−0.431	1.916	0.522	3.15	1.147	23.34	0.043
0.70	−0.357	2.014	0.497	3.20	1.163	24.53	0.041
0.75	−0.288	2.117	0.472	3.25	1.179	25.79	0.039
0.80	−0.223	2.226	0.449	3.30	1.194	27.11	0.037
0.85	−0.163	2.340	0.427	3.35	1.209	28.50	0.035
0.90	−0.105	2.460	0.407	3.40	1.224	29.96	0.033
0.95	−0.051	2.586	0.387	3.45	1.238	31.50	0.032
1.00	0.000	2.718	0.368	3.50	1.253	33.12	0.030
1.05	+0.049	2.858	0.350	3.55	1.267	34.81	0.029
1.10	0.095	3.004	0.333	3.60	1.281	36.60	0.027
1.15	0.140	3.158	0.317	3.65	1.295	38.47	0.026
1.20	0.182	3.320	0.301	3.70	1.308	40.45	0.025
1.25	0.223	3.490	0.287	3.75	1.322	42.52	0.024
1.30	0.262	3.669	0.273	3.80	1.335	44.70	0.022
1.35	0.300	3.857	0.259	3.85	1.348	46.99	0.021
1.40	0.337	4.055	0.247	3.90	1.361	49.40	0.020
1.45	0.372	4.263	0.235	3.95	1.374	51.94	0.019
1.50	0.406	4.482	0.223	4.00	1.386	54.60	0.018
1.55	0.438	4.711	0.212	4.05	1.399	57.40	0.017
1.60	0.470	4.953	0.202	4.10	1.411	60.34	0.017
1.65	0.501	5.207	0.192	4.15	1.423	63.43	0.016
1.70	0.531	5.474	0.183	4.20	1.435	66.69	0.015
1.75	0.560	5.755	0.174	4.25	1.447	70.11	0.014
1.80	0.588	6.050	0.165	4.30	1.459	73.70	0.014
1.85	0.615	6.360	0.157	4.35	1.470	77.48	0.013
1.90	0.642	6.686	0.150	4.40	1.482	81.45	0.012
1.95	0.668	7.029	0.142	4.45	1.493	85.63	0.012
2.00	0.693	7.389	0.135	4.50	1.504	90.02	0.011
2.05	0.718	7.768	0.129	4.55	1.515	94.63	0.011
2.10	0.742	8.166	0.122	4.60	1.526	99.48	0.010
2.15	0.766	8.585	0.116	4.65	1.537	104.58	0.010
2.20	0.789	9.025	0.111	4.70	1.548	109.95	0.009
2.25	0.811	9.488	0.105	4.75	1.558	115.58	0.009
2.30	0.833	9.974	0.100	4.80	1.569	121.51	0.008
2.35	0.854	10.486	0.095	4.85	1.579	127.74	0.008
2.40	0.876	11.023	0.091	4.90	1.589	134.29	0.007
2.45	0.896	11.588	0.086	4.95	1.599	141.17	0.007
2.50	0.916	12.182	0.082	5.00	1.609	148.41	0.007

TABLE VII. SQUARES, CUBES, SQUARE AND CUBE ROOTS

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$		n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$
1	1	1	1	1	**51**	2601	132651	7.141	3.708
2	4	8	1.414	1.260	52	2704	140608	7.211	3.733
3	9	27	1.732	1.442	53	2809	148877	7.280	3.756
4	16	64	2.000	1.587	54	2916	157464	7.348	3.780
5	25	125	2.236	1.710	55	3025	166375	7.416	3.803
6	36	216	2.449	1.817	56	3136	175616	7.483	3.826
7	49	343	2.646	1.913	57	3249	185193	7.550	3.849
8	64	512	2.828	2.000	58	3364	195112	7.616	3.871
9	81	729	3.000	2.080	59	3481	205379	7.681	3.893
10	100	1000	3.162	2.154	60	3600	216000	7.746	3.915
11	121	1331	3.317	2.224	**61**	3721	226981	7.810	3.936
12	144	1728	3.464	2.289	62	3844	238328	7.874	3.958
13	169	2197	3.606	2.351	63	3969	250047	7.937	3.979
14	196	2744	3.742	2.410	64	4096	262144	8.000	4.000
15	225	3375	3.873	2.466	65	4225	274625	8.062	4.021
16	256	4096	4.000	2.520	66	4356	287496	8.124	4.041
17	289	4913	4.123	2.571	67	4489	300763	8.185	4.062
18	324	5832	4.243	2.621	68	4624	314432	8.246	4.082
19	361	6859	4.359	2.668	69	4761	328509	8.307	4.102
20	400	8000	4.472	2.714	70	4900	343000	8.367	4.121
21	441	9261	4.583	2.759	**71**	5041	357911	8.426	4.141
22	484	10648	4.690	2.802	72	5184	373248	8.485	4.160
23	529	12167	4.796	2.844	73	5329	389017	8.544	4.179
24	576	13824	4.899	2.884	74	5476	405224	8.602	4.198
25	625	15625	5.000	2.924	75	5625	421875	8.660	4.217
26	676	17576	5.099	2.962	76	5776	438976	8.718	4.236
27	729	19683	5.196	3.000	77	5929	456533	8.775	4.254
28	784	21952	5.291	3.037	78	6084	474552	8.832	4.273
29	841	24389	5.385	3.072	79	6241	493039	8.888	4.291
30	900	27000	5.477	3.107	80	6400	512000	8.944	4.309
31	961	29791	5.568	3.141	**81**	6561	531441	9.000	4.327
32	1024	32768	5.657	3.175	82	6724	551368	9.055	4.344
33	1089	35937	5.745	3.208	83	6889	571787	9.110	4.362
34	1156	39304	5.831	3.240	84	7056	592704	9.165	4.380
35	1225	42875	5.916	3.271	85	7225	614125	9.220	4.397
36	1296	46656	6.000	3.302	86	7396	636056	9.274	4.414
37	1369	50653	6.083	3.332	87	7569	658503	9.327	4.431
38	1444	54872	6.164	3.362	88	7744	681472	9.381	4.448
39	1521	59319	6.245	3.391	89	7921	704969	9.434	4.465
40	1600	64000	6.325	3.420	90	8100	729000	9.487	4.481
41	1681	68921	6.403	3.448	**91**	8281	753571	9.539	4.498
42	1764	74088	6.481	3.476	92	8464	778688	9.592	4.514
43	1849	79507	6.557	3.503	93	8649	804357	9.644	4.531
44	1936	85184	6.633	3.530	94	8836	830584	9.695	4.547
45	2025	91125	6.708	3.557	95	9025	857375	9.747	4.563
46	2116	97336	6.782	3.583	96	9216	884736	9.798	4.579
47	2209	103823	6.856	3.609	97	9409	912673	9.849	4.595
48	2304	110592	6.928	3.634	98	9604	941192	9.899	4.610
49	2401	117649	7.000	3.659	99	9801	970299	9.950	4.626
50	2500	125000	7.071	3.684	100	10000	1000000	10.000	4.642
n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$

Log$_e$ x, e^x, e^{-x}, n^2, n^3, $\sqrt{n}$, $\sqrt[3]{n}$